Short term joint staff training event: Teachers4STEM

Program

Teaching mathematics in STEM context for STEM students

Project Number: 2019-1-HR01-KA203-060804

LTT Activity C1
Short term joint staff training event: Teachers4STEM
Programme

Day 1, 22nd August 2022

09:30 – 11:00	Biljana Jolevska-Tuneska: The mathSTEM approach of teaching
11:00 – 11:30	Refreshment break
11:30 – 13:00	Nikola Tuneski: Exemplary lessons
13:00 – 15:00	Lunch break
15:00 – 16:00	Round table with discussions

LTT Activity C1
Short term joint staff training event: Teachers4STEM
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Day 2, 23rd August 2022

09:30 – 11:00	Vladimir Radevski: The mathSTEM approach for interactive usage of teaching materials using Wolfram Mathematica
11:00 – 11:30	Refreshment break
11:30 – 13:00	Vladimir Radevski: Exemplary lessons
13:00 – 15:00	Lunch break
15:00 – 16:00	Round table with discussions

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Day 3, 24th August 2022

09:30 – 11:00	Selma Ozcag: STEM Education - Exemplary lesson Emin Ozcag
11:00 – 11:30	Refreshment break
11:30 – 13:00	Hacer Ilhan: Exemplary lesson
13:00 – 15:00	Lunch break
15:00 – 16:00	Round table with discussions

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Day 4, 25th August 2022

09:30 – 11:00	Marjan Praljak: The mathSTEM look at the R
11:00 – 11:30	Refreshment break
11:30 – 13:00	Julije Jakšetić: R+Latex=Sweave
13:00 – 15:00	Lunch break
15:00 – 16:00	Round table with discussions

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Day 5, 26th August 2022

09:30 – 11:00	Neda Lovričević: Descriptive geometry in STEM context
11:00 – 11:30	Refreshment break
11:30 – 13:00	Maja Andrić: Descriptive geometry as a dynamic geometry
13:00 – 15:00	Lunch break
15:00 – 16:00	Round table with discussions

Introduction to STEM methodology
Summer course: teachers4STEM
Biljana Jolevska-Tuneska and Nikola Tuneski

Contents

Chapter 1

Introduction

The increased use of technology in today's education gives the following questions:

- How can technology-supported learning help to move beyond content delivery and truly enhance education so that students develop a broad mix of skills?
- Could innovative and technology supported teaching and learning approaches spark thinking and creativity, enhance student engagement, strengthen communication, and build collaboration?

Trough this paper we will try to give the answers of these questions. We believe that technology-supported education can improve students' learning outcomes, and expand the range of learning opportunities made available to students. Also, students' higher-order thinking – above and beyond content learning – can be fostered by specific technology-supported pedagogic models, and in addition, students' creativity, imagination and problem-solving skills will be enhanced.

Lately, many studies on the effects of the technology supported learning are done. Within these studies both potential advantages and risks from the use of technology are considered and analyzed. Further we will give some examples.

New technology-enhanced educational models present not so much a technological challenge or cost challenge but a pedagogic challenge. To adopt these new models requires teachers to revisit their pedagogy and this may amount to the greatest cost and challenge. The efficacy of the technology-supported models does not come from technology alone, but from the pedagogy that it supports. Without good pedagogic resources and a good understanding of how to use technology to foster deeper learning, these models may not yield the expected outcomes.

Chapter 2

Teaching and learning environments

In this section, traditional and more recent learning arrangements are investigated regarding their potential for competence acquisition. Learning and teaching arrangements appear to have changed little over the years. The predominant form of delivery remains the lecture, and now this is often backed up with supporting materials on a Virtual Learning Environment and may be delivered using much more modern methods than chalk and blackboard.

2.1 Lectures

The most traditional, and probably also the most met form of teaching is the lecture. Lectures can take many different forms. But, in a most known form giving a lecture meant a one-directional presentation of material, during which student activity is primarily restricted to taking notes. Since occasionally a student may ask the lecturer a question, this is considered to be the active component of the lecture, and if more and more students are asking questions, the active learning component is getting stronger. The size of the lecture plays an important role – in smaller lectures (up to 50 students) it is considerably easier to promote an active role of students than in larger ones (in excess of 100 students).

The main reason for lectures is to introduce students to certain concepts and procedures. The goal is to give students a start notions with the material. After that the teacher can introduce an individual or group activities, carried out by the students. These activities are usually necessary to increase understanding of the material. A good lecture should make an interesting approach to the material, make connections with a previous concepts and motivate the students for further work. In a way, the teacher should present the "big picture" of the topics.

2.2 Assignments

Assignments are all kinds of smaller tasks that students have to undertake on their own, be it in groups or individually. These include standard computational tasks that serve to develop more familiarity with notation, formalism and procedures but also more open and investigative assignments, with or without technology.

2.3 Tutorials and projects

Tutorials are learning arrangements where a tutor (teaching assistant or, possibly, a student) works with students in order to improve their understanding related to a lecture. Such tutorials can differ significantly with regard to the method of teaching and learning. There are tutorials where the tutor mainly performs example computations leading to a situation that is not much different from the lecture. But there are also forms of tutorial with active involvement of students who work on standard tasks or on more open assignments with the help of tutors. Students might also give presentations of their solutions on the blackboard.

Projects are learning arrangements where students work (preferably in groups) on problems which are larger, more open and investigative in nature. Usually, students have to document and present their work at the end. In problem-based learning settings this is the way of learning. Thinking, reasoning, problem solving and modeling are all parts of the project based learning.

More realistic problems usually require the use of software, so students also improve their competency of using tools properly. When they create their own experimentation environment for an application situation and try to use it in a goal-directed way, they become accustomed to the way researchers use mathematically-based software programs in their work.

2.4 Technology enhanced learning

Technology also holds significant potential for expanding the range of learning opportunities available to students. The variety of learning opportunities and personalisation technology can offer may make the education more interesting and enjoyable for students.

There are many ways in which technology can be used to enhance the learning process: e-learning, blended learning, on-line learning, etc. This often suggest that the learning activity may be carried out remotely from the presence. Actually, this is the way many schools and Universities worked during the Covid19 pandemic. However, whilst a substantial amount of on-line materials is available, to restrict thinking about enhancing learning only to material for distance learning is an overly narrow view of using technology to enhance learning. As noted in the preceding section, technology can be used to enhance the face-to-face learning experiences.

Materials made for the remote learning the students can access in any time, in order to gain better understanding. This material can be prepared by using lecture capture technology. Animated worked examples can be particularly effective as they allow students to see solutions being developed in real-time (and the audio of the animations allows for explanation of the more difficult steps).

For mathematics, one can find a wealth of supporting material on the web, such as mathcentre (www.mathcentre.ac.uk) and Khan Academy (www.khanacademy.org. These materials provide explanatory text, videos and self-assessments that are not directly related to a specific course but are topic based.

Technology can also enable more interactive learning scenarios: For example, applets or other small learning objects can be produced which allow students to make changes (e.g. parameter variations) and determine their effect, or to work on tasks to achieve certain properties by making variations. There are also more sophisticated intelligent tutoring systems which allow the insertion of single steps and provide tutorial help. An electronic forum might also be used as a means of collaboration and communication between students, or between students and tutors/lecturers. More recently there has been the advent of massive open online courses as part of the open educational resource movement. Some universities in the US and elsewhere have partnered with companies, to make some of their courses freely available online to a large audience.

Technology-enhanced learning offers competence acquisition opportunities similar to the one we discussed before. But there are risks involving the passive side of the communication competencies. This item can be solved if students joint groups and discuss problems together.

Chapter 3

Mathematics technology

Advances in the capabilities and user-friendliness of mathematical software mean that a whole range of problems which previously would have needed graduate level skills to solve can now be accessed by first year undergraduates. For example, to solve an integral in *Mathematica* these days is enough to know the Integrate command, and the syntax in this software. There is no need for complex rules of the integration process. The question that arise is: What we consider to be a technology for mathematical education, or "mathematics technology?"? In this set we can put the following items:

- Pocket calculators with different symbolic and/or numerical and/or graphical capabilities.
- Mathematical computer programs: symbolic and numerical ones, e.g. Computer Algebra Systems (CAS) like Maple, Mathematica, or MathCad;
- Numerical programs like Matlab.
- Dynamic geometry programs like Cabri and Geogebra.
- Spreadsheet programs.
- Engineering programs based on mathematical models.

The effect of computer technology on education seems to be greater in mathematics than in any other subject. There are two distinct ways in which developments in technology affect learning and teaching in mathematics.

- New technology provides opportunities for new approaches to teaching and learning.
- Advances in technology impact not only on how mathematics is taught but also on what mathematics is taught.

Modern technology also creates some new challenges and tasks in the mathematical education that must be addressed for technology to really improve teaching and learning:

- One large new challenge is to strike the right balance between work with and without technology in a blended learning approach. Some participants advocated an approach where concepts come first and their application using technology comes later.
- Learners should discuss and experience an adequate use of technology. They should for example see that there is no need for technology when it comes to computing $\cos(0)$. Students should also experience the limitations and pitfalls of mathematical technology (for example: are all possible cases covered in a computational procedure?) such that they develop a critical attitude towards technology and can make goal-oriented use of technology.

Technology will be there and cannot be ignored.

We recognize the following new opportunities and potential advantages provided by technology:

- Better visualization of mathematical concepts by using mathematical programmes.

  In some area of mathematical education the visualization and demonstration is very important for the effectiveness of the teaching process, for example, in calculus or multivariable calculus.

- Explorative approach to learning.

  Using technology, the technical dimension of mathematical activities are been facilitated. It allow the user to take action on mathematical objects or representations of those objects. This feature can be utilised to enable students to explore objects and structures and to discover properties and connections e.g. by performing parameter variations.

- Experimental approach to problem solving.

  Mathematics programs provide new ways of problem solving. In classical paper and pencil work students had to know a certain procedure in order to solve a problem and they could not advance once they got stuck in the process. Mathematics programs allow students to select different ways of investigating a problem (for example finding an approximate solution by looking at the graph of a function instead of getting an exact solution; investigating several related examples to derive a hypothesis or to discover a counter-examples and thus increase the students chances in making a progress with a problem.

- Realistic modeling.

  Mathematics and engineering programs allow the earlier introduction of more interesting modelling tasks since some parts of the computation can be delegated to the program.

- Change of roles.

  Using mathematics programs can help to bring about changes in the way classes are conducted, as their usage requires student active participation and bigger activity. Such activity can also be designed to require interaction among students. The result is that the process of acquiring and developing mathematical knowledge becomes more student-centred. This also changes the role of teachers, who become tutors and instructors rather than lecturers.

- Large data manipulation in more realistic projects.

  Experiential learning is most likely to provide expected improvements in conceptual understanding and scientific inquiry skills if teachers encourage students to repeat their experiments and provide students with a robust scaffolding to understand them.

- Easier assessment.

  Real-time formative assessment (with a help of the technology) allows teachers to see in real time what students think and know, but they still have to use this information in their teaching to encourage students to reflect more deeply and to challenge their misconceptions.

- Use of chat facilities.

  The use of the chat facility allowed students to participate more actively in lectures by posing questions or giving answers to questions posed by the lecturer. Why students are more comfortable with the chat option rather than asking the teacher directly is yet to be investigated. Maybe is because whereas in class only one student gives an answer, the chat allows for many students to give answers simultaneously. Moreover, the chat seemed to constitute a smaller hurdle for asking questions than the classroom situation. The question is how this can be transferred to normal class teaching. Even if the technology is available, it is hard for a lecturer to give the lecture and at the same time to observe the chat.

- Use of smartphones for activating students by voting.

  Polling or voting systems are also a means to involve students more. The polling can be offered via smart phones.

- Usage of videos for flipped classroom scenarios (lecture becomes tutorial).

  Many short video clips or longer recorded lectures are now available. These digital offerings increase the flexibility of learning and were hence appreciated by students.

- Motivational aspects.

  Most students are accustomed to using technology such as smart phones in their daily life; consequently simply having technology involved can make a huge difference in students attitudes and feelings towards mathematics. Therefore increased use of mathematics technology may help to improve student motivation for learning.

For sure, the use of technology in the mathematical education has its cons. The following risks have been identified:

- Loss of basic capabilities.

  When adapting the mathematical educational process to make use of new technological tools one must be aware of the risk that this computer-based learning environment may cause an unexpected reduction of students knowledge in basic "mathematical culture". This is not just a loss of fluency in carrying out procedural mathematical tasks brought about by a reduced amount of practice (due to using computer programs to carry out these tasks), but can also be a more limited understanding of core mathematical concepts as the reduced practice may bring with it reduced need to think about the basic concepts thereby impacting on overall mathematical reasoning skills.

- Loss of connection between procedures and understanding.

  Extensive and exclusive usage of mathematical technology can potentially prevent students from making proper connections between the techniques used for calculations and conceptual understanding, for example the Gauss algorithm for solving a linear system of equations also provides insight into the possible solution types; simple use of "solve" command does not give this insight.

- Pure trial and error working style without thinking.

  There is a danger that students may use mathematics and application programs in a largely thoughtless trial and error mode, making variations without any particular strategy in the hope that somehow they will achieve what is required without having any idea why what they are doing. Problems must be found where such a strategy is not productive so that students are forced to think about the effects of possible variations.

- Tool dependence.

  When students are no longer able to compute even simple examples by hand, they depend totally on what the tool they are using provides. They also have no idea of what to do when a program fails to give them an answer to a problem because they do not know what the program is able to do. Although the students do not need to know in detail what a program does, they should know which model a program is appropriate to use or not appropriate.

3.1 Mathematics Laboratories

What is mathematics laboratory? Can we speak these days about Mathematics Laboratories?

By mathematics laboratories, we mean learning scenarios where students work in a PC laboratory on tasks requiring the use of mathematical software such as numerical programs Matlab, Maple, Mathematica or spreadsheets Excel. In such laboratory sessions, students practise the usage of the programs and see how they can be used for standard tasks. They might also be used for experimenting with more open tasks of an investigative nature.

3.2 Assessment

in this section we will give the answer on the question: How can the students knowledge be evaluated?

Assessing and grading are extremely important parts of the teacher's work. The grade achieved by a student, in relation to what other students have achieved, can determine his/her future, the first job, a PhD education for instance. The students know this and find it in general extremely annoying – it may even have a strong negative impact on the interest in the subject – if the assessment is considered unfair or if it seems to be safe to cheat to get a better grade.

We have several forms of the assessment:

- Written examination.

  The most common assessment method is a written examination, with closed books, at the end of the course. Less common is a written examination with open books or computer facilities to support the problem solving. One argument for allowing computers and/or advanced calculators is that the assessment situation should be as realistic as possible, meaning that in future the students will (of course) use every suitable technology to solve a real life problem.

  However, modern advanced electronic equipment can communicate wirelessly over long distances. Therefore, we cannot claim legal certainty in these assessment situations unless other assessment methods, such as oral presentations, are added.

- Take-away assignments.

  Take-away assignments are used at several institutions, but always as one amongst a number of methods of assessment and never as the only or primary method. They give students an opportunity to explore more realistic problems than they can in an ordinary written examination and for this reason often require the use of computer software to complete the assessment task. Some teachers have reservations about this method of assessment because it is impossible to be certain that the student submitting the work actually did it for him/her-self. When the take-away assignment is followed up with an oral presentation of the work the legal certainty is stronger.

- Multiple-choice tests.

  Only a few institutions use multiple-choice tests and those that do use them do so only occasionally. Such tests can be cheap to administer as they can be computer delivered and marked. They can be useful in giving formative feedback during the course. There are reasons to believe that the use of this kind of computer-supported assessment is increasing.

Other methods of assessment such as project work, group work and oral presentations are not widely used. However, when it comes to examination of mathematical competencies, these methods can be more interesting. It is difficult to give individual grading of group work, but individual timelogs, progress-logs and contribution reports together with the project report can support the grading.

Chapter 4

STEM education

What should STEM education look like now and in the future? This section presents five innovative models of technology-supported STEM teaching and learning. These are educational gaming, online laboratories, collaboration through technology, real-time formative assessment and skills-based curriculum alignment. The emphasis is on practices that would be difficult to implement without technology and that can improve not just traditional learning outcomes, but also motivation, social, behavioural, thinking and creativity skills and their assessment.

4.1 Educational gaming

Educational gaming offers a promising model to enhance student learning in STEM education, not just improving content knowledge, but also motivation and thinking and creativity skills. Educators and policy makers should consider using it to enhance STEM learning outcomes and problem-solving skills and motivation. Designing games appears to lead to even deeper learning than just using them for educational purposes.

In educational gaming students interact with video games, simulations or virtual worlds based on imaginary or real worlds, also seen as highly interactive virtual environments. Educational gaming also includes collaborative project-based learning experiences where students themselves become game designers and content producers.

As a promising model for various disciplines and education levels, educational gaming may promote:

- Learning by doing.

  The interactive, reactive and often collaborative nature of educational gaming enable learning by doing of complex topics by allowing students to (repeatedly) make mistakes and learn from them. Real-life based gaming allows experimentation that would otherwise be too costly or dangerous.

- Student learning.

  Educational gaming which covers specific topics or subject areas and take place within a set of rules can increase students' achievements and subject-specific knowledge. Constructing educational games seems to increase deep learning more than just using existing games.

- Student engagement and motivation.

  Based on play and increasing challenges, educational gaming can foster student engagement and motivation in various subjects and education levels. Low-achieving students may find the educational gaming experience more engaging than high achieving students. Students' motivation can increase more when they construct games themselves as opposed to just playing an existing game.

- Students' thinking skills.

  Games have the potential to help students find new ways around challenges, use knowledge in new ways and "think like a professional". Educational gaming may also improve students' skills such as problem solving.

Example 3. One example of educational game is the GRADIENT DESCENT, one can find on https://www.i-am.ai/gradient-descent.html.

Trough this game the player should find the treasure hidden in the deepest point of the ocean floor by lowering a probe from the research ship. Players use the arrow buttons to move the ship and the circle button to send the probe.

After playing a couple of times player should probably figured out that the quickest way to find the deepest point in the ocean floor is by paying attention to the slope found by the probe: how steep the floor was at that point and in which direction it was inclined. Although the player can't see the bottom and don't have a full picture of what it looks like, the slope suggests where to continue the search.

The best treasure searching strategy is the an algorithm called (just like our game) gradient descent.

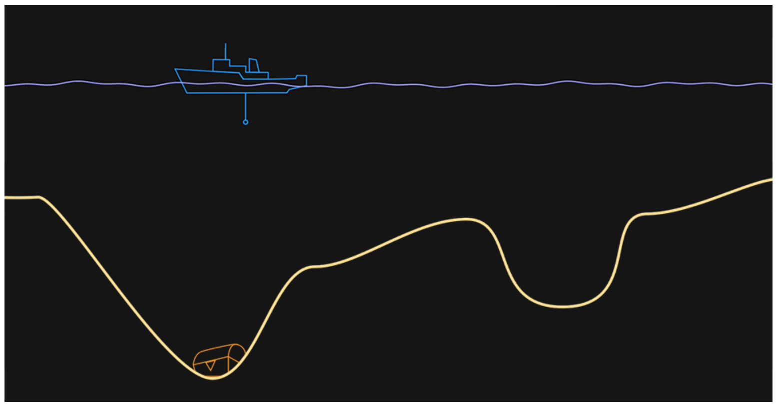

Figure 4.1:

If you start searching over the left side of the screen you'll find the deepest point, where the treasure is. If you start searching over the right side you might find the local minimum instead.

4.2 Online laboratories

Online laboratories, whether remote or virtual, are another promising innovation to enhance the teaching and learning of STEM at all levels of education. Virtual online laboratories allow students to simulate scientific experiments while remote ones allow students to use real laboratory equipment from a distance through the Internet.

Educators and policy makers should consider online laboratories as a promising way to increase access to a wide range of experimental learning. The use of online laboratories only requires access to the Internet and allows teachers and students to get access to more experimental equipment than a single school can generally provide. While remote laboratories can give students access to expensive equipment, virtual laboratories can allow them to vary the conditions for the experiments. Online laboratories are thus a good complement to – or substitute for – school science labs. The use of online laboratories can be at least as effective in terms of learning as the use of on-site physical equipment, and many resources are freely available on the web.

As promising innovations especially for science instruction, online laboratories can (be expected to) offer the following potential benefits:

- Lower-cost access.

  Online laboratories may help bridge the digital divide by providing students with faster access to experimental learning at a relatively low cost. Simulations may be less expensive than experimental hardware, although "little empirical data exists on the actual costs of providing online laboratory access at scale"

- Flexible access.

  Online laboratories can enable flexible access to practical experiments, allowing increased study time that is not tied to a specific timetable or location.

- Student learning.

  Online laboratories can help support student understanding and achievement at least as well as physical hands-on learning does. Virtual environment may be used in a blended format together with physical environment of experimentation to further increase student understanding.

4.3 Collaboration through technology

Collaboration through technology can enhance students' interaction, engagement, learning and thinking skills, in addition to increasing flexibility and diversity in educational experience. Technology-supported collaboration can enhance students' awareness of global challenges and develop their understanding of other cultures.

Example 4. Two students groups from different countries can examine the water quality of the rivers in their school towns. Students can measure water quality, topography, drainage, flora and fauna, as well as the impact of urban development on water quality. With the help of technology, the joint projects of this type enabled students in both countries to compare their findings and reflect on the challenge of water quality internationally. Their awareness of another culture will be increased too.

In technology-enabled collaboration, students work together (in groups) and/or interact with each other to enhance their learning with the help of various technologies and often with facilitation from the teacher. When combined with other learning approaches, technology-enabled collaboration can form a part of project- or problem based learning or supplement face-to-face learning. Technology-enabled collaboration models may include in-built assessment features taking into account also team performance and/or collaborative activity.

As a promising model for STEM education and other disciplines at various education levels, collaboration through technology may improve:

- Flexibility.

  Technology enables students to collaborate and practice at "their own pace", beyond the formal classroom hours and without limitations of physical location.

- Cultural diversity.

  Technology can significantly increase possibilities for intercultural interactions by broadening the scope of collaborations to distant locations, even across borders.

- Student learning.

  Technology-enabled collaboration may support student learning, in both individual and group outcomes, although not necessarily more than face-to-face interaction. There can also be cross-cultural differences. In general, positive results of co-operative learning on student achievement have been shown to depend on group learning goals and individual accountability.

- Student interaction and engagement.

  Technology-enabled collaboration can encourage student group work skills, interaction and engagement. Yet, "active learning strategies" are not automatically adopted and activity may differ across cultures. In general, co-operative learning has shown clearly beneficial results on affective student outcomes.

- Students' thinking skills.

  Online collaboration may enhance higher order thinking even more than face-to-face collaboration through "more complex, and more cognitively challenging discussions". This can also be the case for "questioning behaviours" and "project performance".

4.4 Real-time formative assessment

Technology significantly facilitates the use of formative assessment - that is, frequent, interactive assessment of student progress and understanding. Clickers, tablet computers and other kinds of technology enable instantaneous interaction and feedback between teachers and students. In real-time formative assessment, software enables a variety of inputs to be used for student assessment including open format replies, student questions, pictures or mathematical formulas. Some of the software is freely available. Real-time formative assessment can be combined with various instructional models.

As a promising educational innovation, real-time formative assessment could enhance:

- Targeted instruction.

  Real-time formative assessment allows teachers to monitor student learning as it happens and better adjust their teaching to the needs of individual students.

- Student learning.

  Real-time formative assessment can increase student achievement by promoting students' reflection about the needs of and engagement in their own learning.

- Problem solving and creativity.

  Real-time formative assessment provides avenues for assessing different types of activities and variety of student skills such as problem solving or creativity – potentially enhancing the acquisition of these skills.

Chapter 5

Exemplary lessons: Vector fields

Opening problem:

Hurricanes are huge storms that can produce tremendous amounts of damage to life and property, especially when they reach land. Predicting where and when they will strike and how strong the winds will be is of great importance for preparing for protection or evacuation. Scientists rely on studies of rotational vector fields for their forecasts. Shown in Figure 5.1

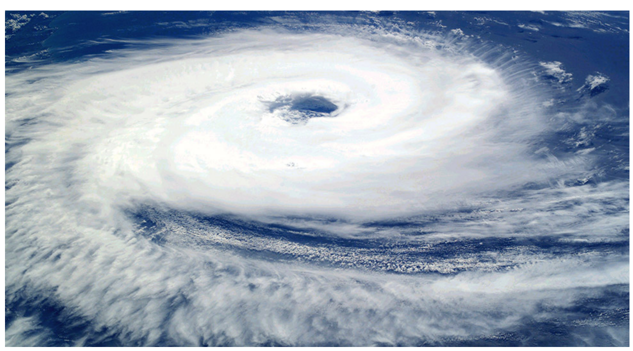

Figure 5.1:

is Cyclone Catarina in the South Atlantic Ocean in 2004, as seen from the International Space Station.

These applications are based on the concept of a vector field, which we explore in this chapter. Vector fields have many applications because they can be used to model real fields such as electromagnetic or gravitational fields. A deep understanding of physics or engineering is impossible without an understanding of vector fields. Furthermore, vector fields have mathematical properties that are worthy of study in their own right. In particular, vector fields can be used to develop several higher-dimensional versions of the Fundamental Theorem of Calculus.

Examples of vector fields. How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river? Figure 5.2 gives visual representations of such phenomena. 5.2 a shows a gravitational field exerted by two

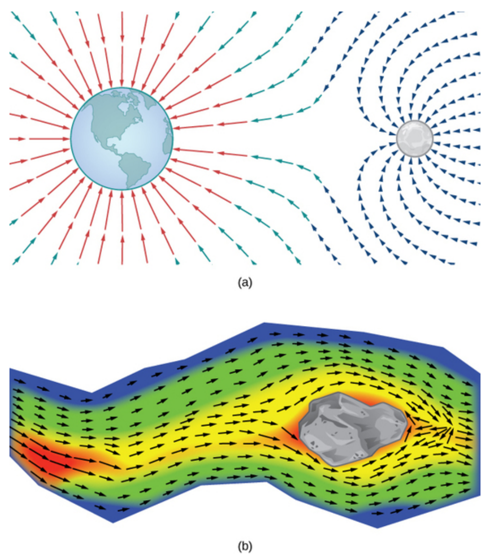

Figure 5.2:

astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

Figure 5.2b shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.

Each figure illustrates an example of a vector field.

Definition 5.0.1. *A vector field in* $\mathbb{R}^3$ *is a function that assigns to each point* (x, y, z) *a vector in space* $\vec{F} = \vec{F}(x, y, x)$ *. The standard notation for the* $\vec{F}$ *function is:*

$$\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k},$$

where M*,* N *and* P *are differentiable scalar functions.*

Example 5.0.1. *Sketch each of the following vector field* $\vec{F}(x, y, z) = 2x\vec{i} - 2y\vec{j} - 2x\vec{k}$ *.*

We made the sketching in $Mathematica$.

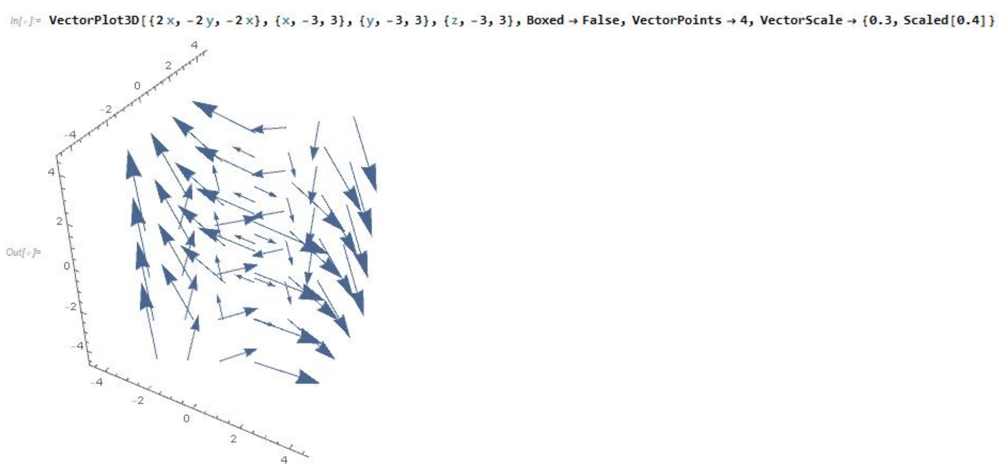

Figure 5.3:

Example 5.0.2. *Vector field* $\vec{v} = 4|x|\vec{i} + \vec{j}$ *models the velocity of water on the surface of a river. What is the speed of the water at point* $A(2, 3)$*? Use meters per second as the units.*

The vector field $\vec{v}$ at $A(2, 3)$ will be $\vec{v}_A = 8\vec{i} + \vec{j}$. The speed of the water at this point is the magnitude of this vector, i.e., $speed = |\vec{v}_A| = \sqrt{8^2 + 1^2} = \sqrt{65} = 8.06 m/s$.

Describing a Gravitational Vector Field.

Newton's law of gravitation states that $\vec{F} = -G\frac{m_1 m_2}{r^2}\vec{r}$ where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1) of mass m_1 located at the origin on another object of mass m_2 located at point (x, y, z). Field $\vec{F}$ denotes the gravitational force that object 1 exerts on object 2, r is the distance between two objects, and $\vec{r}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. We will sketch the vector field associated with this equation.

We locate the object 1 in the origin. Now the distance between object 1 and object 2 is $r = \sqrt{x^2 + y^2 + z^2}$, and the unit vector is $\vec{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$. Therefor the gravitational vector field is:

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\vec{r} = -G\frac{m_1 m_2}{r^2}\left(\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right) = -G\frac{m_1 m_2}{r^3}\left(x\vec{i} + y\vec{j} + z\vec{k}\right). \tag{5.1}$$

On the Figure 5.4 the gravitational vector field is given, sketched in *Mathematica*. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.

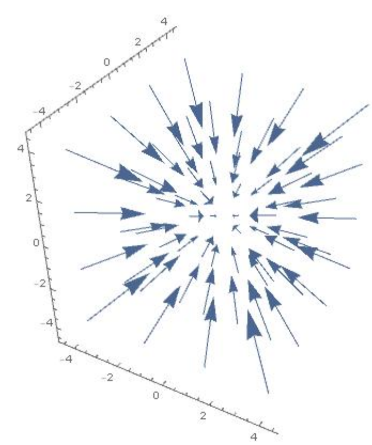

Figure 5.4:

Example 5.0.3. *The mass of asteroid 1 is* 750,000 *kg and the mass of asteroid 2 is* 130,000*kg. Assume asteroid 1 is located at the origin, and asteroid 2 is located at* $(15, -5, 10)$ *, measured in units of 10 to the eighth* *power kilometers. Given that the universal gravitational constant is* $G = 6.67384 \times 10^{-11} m^3 kg^{-1} s^{-2}$*, find the gravitational force vector that asteroid 1 exerts on asteroid 2.*

Gradient Vector Field. On of the "famous" vector field for a scalar function f is its gradient vector field,

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\,. \tag{5.2}$$

These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Example 5.0.4. *Find the gradient vector field for the function* $f(x, y, z) = ze^{-xy}$.

Using (5.2) we have:

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = -yze^{-xy}\vec{i} + -xze^{-xy}\vec{j} + e^{-xy}\vec{k}\,.$$

Divergence and Curl

In this section, we examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. For example, If $\vec{F}$ represents the velocity of a fluid, then the divergence of $\vec{F}$ at point P measures the net rate of change with respect to time of the amount of fluid flowing away from P (the tendency of the fluid to flow "out of" P). In particular, if the amount of fluid flowing into P is the same as the amount flowing out, then the divergence at P is zero.

One application for divergence occurs in physics, when working with magnetic fields. Physicists use divergence in Gauss's law for magnetism which states that if $\vec{B}$ is a magnetic field, then its divergence is zero.

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. In other words, the curl at a point is a measure of the vector field's "spin" at a point.

Definition 5.0.2. *Let $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a vector field, where M, N and P are differential functions. The divergence of $\vec{F}$ is the scalar function*

$$div\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z},$$

and curl of $\vec{F}$ is the vector field

$$curl\vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}.$$

We define an operator ∇ with

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.$$

Note that the divergence of $\vec{F}$ is the dot product of ∇ and $\vec{F}$:

$$\text{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (M\vec{i} + N\vec{j} + P\vec{k}) = \nabla \cdot \vec{F}.$$

The curl of $\vec{F}$ is vector product of ∇ and $\vec{F}$: ,

$$\begin{aligned} \text{curl}\vec{F} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k} \\ &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times (M\vec{i} + N\vec{j} + P\vec{k}) = \nabla \times \vec{F}. \end{aligned}$$

Example 5.0.5. *Find the Divergence and Curl of the vector field $\vec{F}(x,y,z) = x^2y\vec{i} + xyz\vec{j} + (y^2 + z^2)\vec{k}$.*

We have:

$$\begin{aligned}
\operatorname{div}\vec{F} &= \nabla \cdot \vec{F} \\
&= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \cdot (x^2 y \vec{i} + xyz\vec{j} + (y^2 + z^2)\vec{k}) \\
&= \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(y^2 + z^2) \\
&= 2xy + xz + 2z\,.
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl}\vec{F} &= \nabla \times \vec{F} \\
&= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \times \left(x^2 y \vec{i} + xyz\vec{j} + (y^2 + z^2)\vec{k}\right) \\
&= \left[\frac{\partial}{\partial y}(y^2 + z^2) - \frac{\partial}{\partial z}(xyz)\right]\vec{i} - \left[\frac{\partial}{\partial x}(y^2 + z^2) - \frac{\partial}{\partial z}(x^2 y)\right]\vec{j} + \\
&\quad + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(x^2 y)\right]\vec{k} \\
&= (2y - xy)\vec{i} + (yz - x^2)\vec{k}\,.
\end{aligned}$$

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SEEU MathSTEM teaching methodology

Teachers' training, Split 21-26.08.2022

Teaching mathematics in STEM context for STEM students

Experiences and contribution on behalf of SEE University,

Tetovo, Republic of North Macedonia

Lecturers: Vladimir Radevski, Murat Sadiku, Halil Snopce

Content:

1. BACKGROUND AND INSTITUTIONAL CONTEXT

At South-East European University there is a variety of mathematical courses offered on various level of studies and study programs. They are delivered at the academic units: Faculty of Contemporary Sciences and Technologies (CST) and Faculty of Business and Economy (FBE) on the study programs for 1st cycle of studies (Undergraduate programs) Computer Sciences and Engineering, Business Informatics in the former, and Business and Economy in the later unit.

The mathematical topics are delivered in the framework of the following courses: Calculus, Discrete Mathematics, Probability and Statistics and Business Mathematics.

In this project we have been focused on the tools and manners of delivering and facilitating mathematics topic delivery on undergraduate level both in fundamental and applied mathematics and in the framework of the existing courses. We have linked our efforts to already existing practices in usage of IT support in teaching including the usage and activities going on in our STEM Laboratory.

2. Experience and considerations

We have considered a variety of course delivery components in usage at our institution and at institutions members of the consortium of this project. Courses are described through a variety of syllabi formats, mainly reading documents and have been delivered using various supports as Power Point presentations available for students, pre-recorded lessons (especially during covid-19 crisis) links to portals, YouTube materials and specialized tools mainly in animations and automatic calculations. There is a wide variety of technologies and formats in usage through different platforms present at all partners of the project as Wolfram Mathematics, E-textbooks, MS Office standards, Adobe products to name a few.

Course management systems are also widely in usage as Google Classroom, Moodle and similar. They are offering posting options for the above-mentioned materials, and mostly limited options for interaction mainly based on forums of messages and upload options for students' individual work.

3. Students' perspective and needs identification

We have been following the students' practices and usage of software and tools during their learning. We have also performed a questionary of the habits and needs of the students during learning process of mathematical courses. The questionary showed that the students extensively use popular software during the learning process and mainly the following applications: PhotoMath, Symbolab, Wolfram suite, Maple, Graphing calculators (Desmos), Matlab.

As far as needs identified, great majority of the students expressed in the first place the need of having a possibility to show the solving techniques in details and in a step – by – step manner with explanations.

In conclusion, and based on the feedback received form the students we have concluded that all previous tools and methods present as teaching and learning support tools already present in our academic practice of delivering mathematical courses lack of interactivity and that is the need that we have focused to cover in our work. Offering interactive environment for supporting the learning process will address at least two identified needs: 1) On-hand experience during learning process to cover the step by step approach both as approach but also with ability to interact at each specific stage of the solving process and 2) Especially for the real-world problems of application of mathematical concepts this will significantly improve the clearness and the meaningfulness od the covered topics and techniques. The later is especially important for the courses in STEM domain, but also in our institution in applied mathematics in Business and Economy since the application exercises can be often overwhelming with unclear correlation of the mathematical methods in use and the real-word problem representation and solution techniques.

We have addressed these issues on the example from the application of Definite integral in real-world problems solving in Business and Economy and by offering synchronous interactivity through the mobile devices of the students using Wolfram Mathematics Cloud tools.

In summary the aims behind this study were to:

- Adopt personalized solutions in a variety of study programs and mathematical learning aims
- To address the need for advanced academic approach in shaping mathematical topics delivery according to students' level and program needs
- Contribute to a smart usage of available online resources (portals, videos, applications, tutorials) to support STEM students in learning Mathematical subjects
- Offer hands-on experience and interactivity both during learning a mathematical concept and during experimenting with real- world applications of mathematical theory in the area of Computer Sciences and Business and Economy.

Ultimately, we contribute towards supporting mathematical thinking, abstraction, analysis and reasoning with the support of IT based solutions.

4. Wolfram Mathematics potentials and usage

As defined on the portal of Wolfram suite of applications, Wolfram Mathematica " … for three decades of existence, has defined the state of the art in technical computing—and provided the principal computation environment for millions of innovators, educators, students and others around the world." (http://www.wolfram.com)

Starting from its characteristics to be "Widely admired for both its technical prowess and elegant ease of use, Mathematica provides a single integrated, continually expanding system that covers the breadth and depth of technical computing—and is seamlessly available in the cloud through any web browser, as well as natively on all modern desktop systems." We will focus in this project on the recent advances of manipulation and availability tools present and offered on the cloud.

All applications are developed in the standard Wolfram Mathematica format of "*notebooks*" an interface containing code, text, images and graphics. These runnable pieces of code since recently where available for running only in the standard WM environment, but since the beginning of 2020 the WM package versions offer runnable versions of the *notebooks* available on the WM cloud and through simple automatically generated QR code in the framework of the package instantaneously accessible in runnable format on mobile devices.

Moreover, a rich well-presented set of “demonstrations” on a wide variety of STEM topics is available through WM site and the correspondent archive of *notebooks*.

We are presenting these *demonstrations* and *notebooks* in the following link: https://demonstrations.wolfram.com/

EXAMPLE: Area between curves

An example of such demonstration instantaneously available to be shared in a synchronous manner on students’ mobile devices is the following example on the Area between curves problem:
https://demonstrations.wolfram.com/topic.html?topic=Integrals&start=21&limit=20&sortmethod=recent

This example is instantaneously available to be run on a personal mobile device through the following automatically generated QR code:

QR code for access to the Area between curves demonstration

from the WM package.

It worth to note that this ability a *notebook* to be offered on the cloud is valid for all kind of specialized notebooks, for example a notebook for simple slide presentations. This is the case for the following example.

EXAMPLE: Using Notebooks for simple presentations

with the following QR code:

An example of a simple slide presentation done with a notebook setting

in the presentation's module of Wolfram Mathematica.

The correspondent slide presentation generated with WM accessible through this code on a mobile device will display the following message:

This is a message that

Was just developed in front of you and it is instantaneously available on your mobile devices...

Enjoy

-

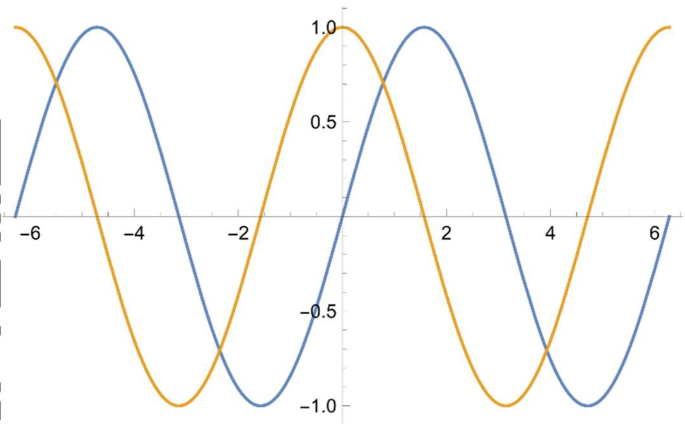
A simple transfer of photo and textual message to students' mobile devices.

EXAMPLE: Simple one-line code notebook for instant demonstrations

The following is an example of a one – line code notebook for experimentation with the trigonometric functions:

Notebook:

```
Plot[{Sin[x], Cos[x]}, {x, -2Pi, 2Pi}]
```



An example of simple notebook code for experimentation with trigonometrical functions.

The above presented examples show easily transferable examples, demonstrations, simple code notebooks and presentations generated with WM package, Published on WM Cloud and accessible through automatically generated QR code for that purpose. In classroom these examples are immediately accessible on students' mobile devices which is an excellent opportunity for offering common working space and portability of the developed examples.

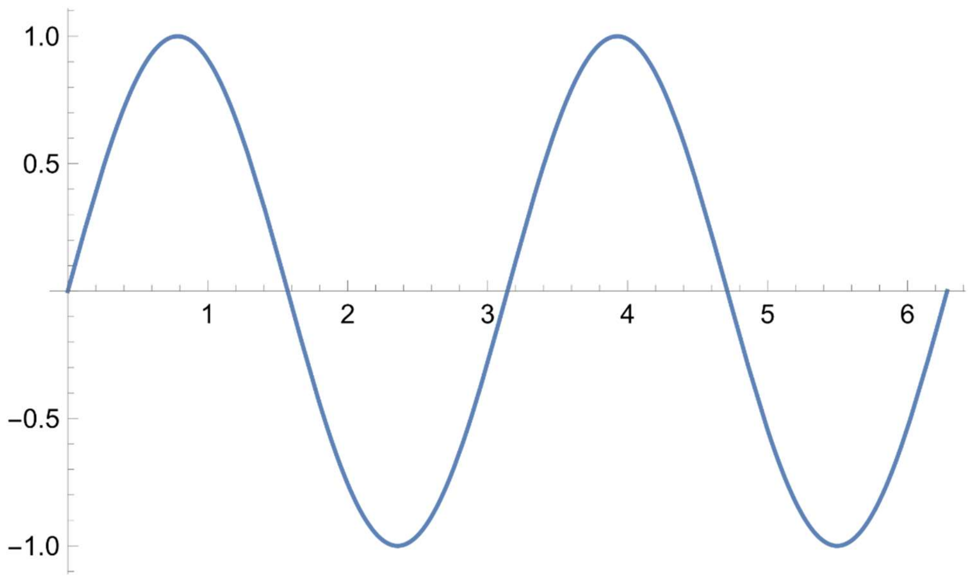
Further on, we examine simple interactivity examples based on simple notebooks in Wolfram Mathematica including the interactive option most frequently based on buttons, bars, multiple choice tabs and similar manipulation tools adding to the hands-on experience exclusive experience in students' own mobile devices.

EXAMPLE: Introducing interactivity in notebook manipulation

Interactivity in WM notebooks can be easily introduced as it is shown in the following example:

Y = Sin (nx)

```
Manipulate[Plot[Sin[n x],{x,0,2
Pi}],{n,1,20,Appearance->"Labeled"}]
```



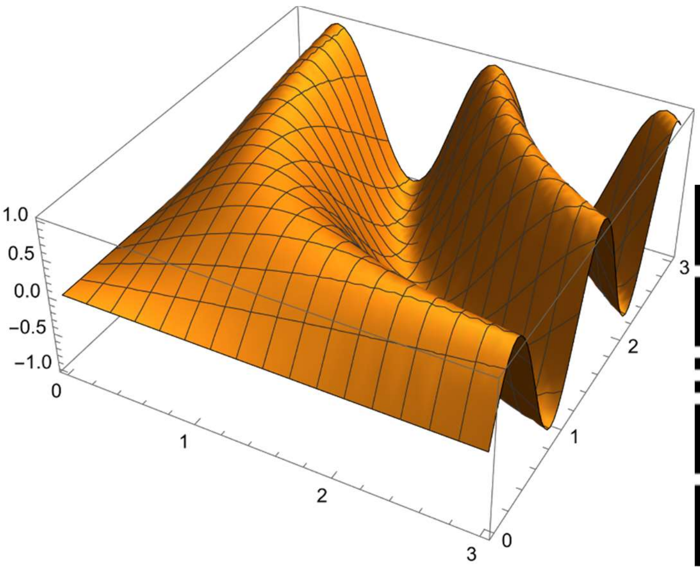
This example published to Wolfram Cloud has the following QR code automatically generated:

The conde for the first interactive examples y = sin nx

EXAMPLE: 3-d plotting

Notebooks concept in WM allows easy plotting, shown with the following on-line code for plotting y = sin nxy

```
Manipulate[Plot3D[Sin[n x
y],{x,0,3},{y,0,3}],{n,1,5}]
```



3-d plotting example and the correspondent QR code

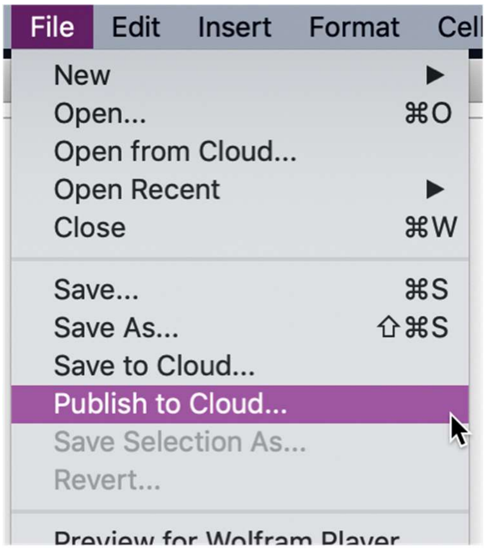
All previous examples were published to WM Cloud using the simple option from the notebooks editing interface as shown below https://www.wolfram.com/language/12/notebook-interface/publish-notebooks-to-the-wolfram-cloud.html?product=mathematica

:

Publish Notebooks to the Wolfram Cloud

Using a Wolfram Cloud account, you can publish any notebook in a public location and send out the URL so that anybody on the web can read and interact with the notebook. Users will not be able to change your published notebook, but they will be able to interact with Manipulate outputs, download the notebook or copy it into their own Wolfram Cloud accounts.

Use **File ▸ Publish to Cloud...** to publish the selected notebook interactively.



Publishing notebooks to WM Cloud

5. Recommendations and case study on course development using WM tools

In the framework of the course Calculus delivered for Business and Economy students with mandatory real – world examples component we have developed a WM notebook for solving a practical problem in the domain of Definite Calculus.

Here is the exercise:

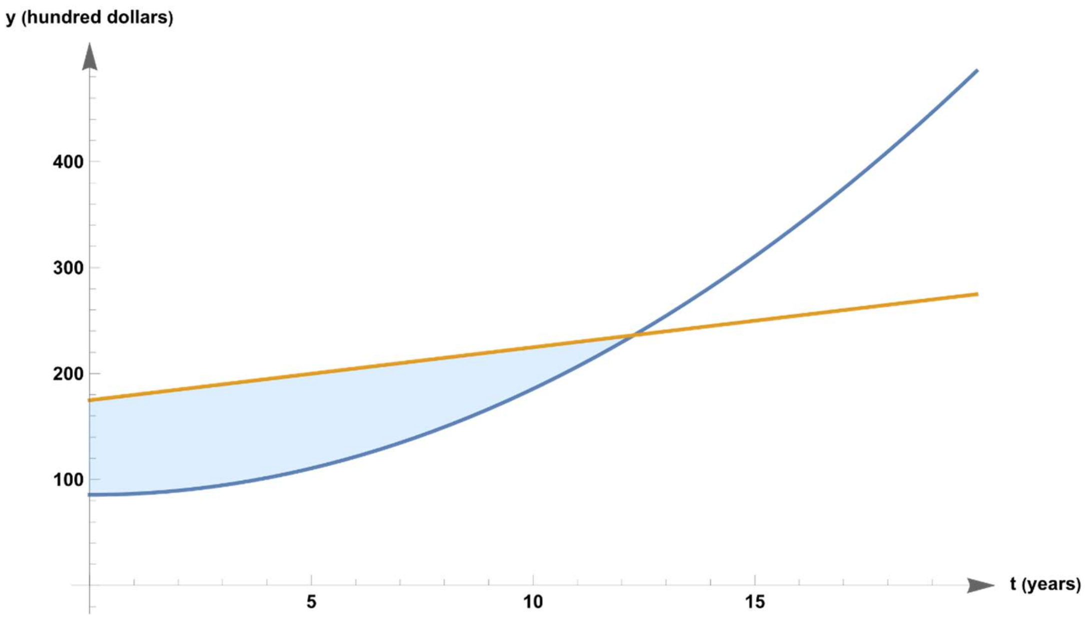
Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = 50 + t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t) = 200 + 5t$ hundred dollars per year.

a. For how many years does the rate of profitability of the second investment exceed that of the first?

b. Compute the net excess profit for the time period determined in part (a). Interpret the net excess profit as an area.

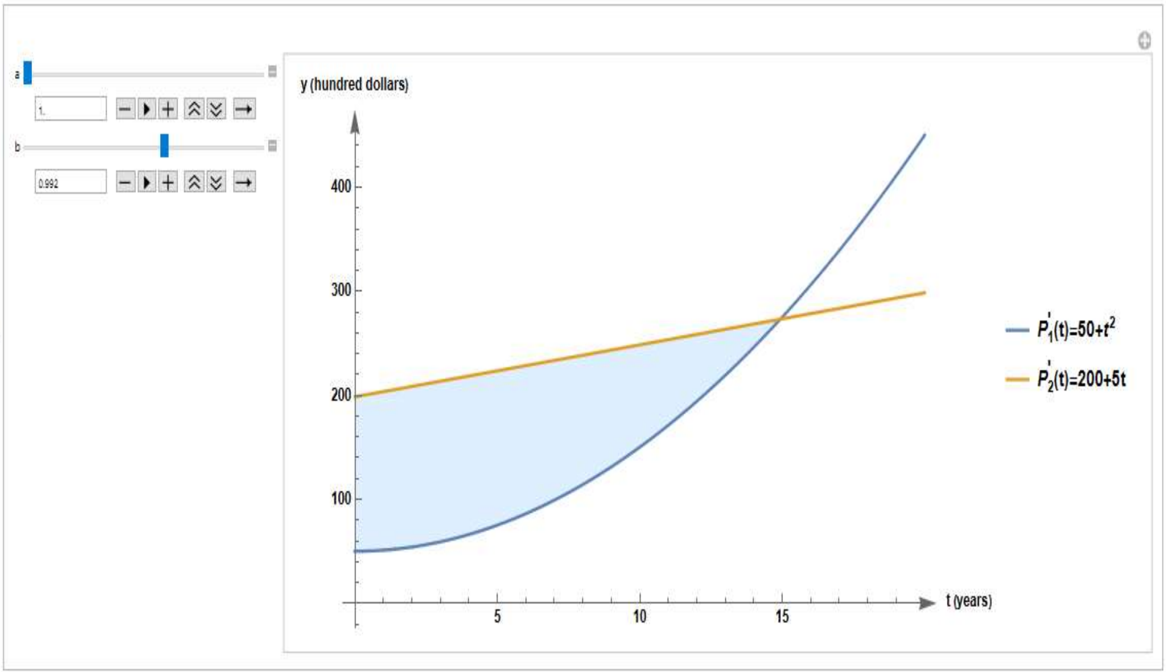
(Example taken from Title: CALCULUS Author: Laurence D. Hoffmann, Gerald L. Bradley Publisher: McGraw-Hill,Year: 2000)

Rate of profits are given with the correspondent derivative functions and the first step is to allow to the students to see the plot of the two functions and to allow them “playing” with functions’ graphs obtained by changing constant values in the functions’ expressions:



Showing the plot of the graphs of the functions involved in the exercise

Showing controls in the WM notebook on desktop on students’ mobile devices:



This experience will allow to the students to get "in – touch" with the intersection point of the two functions (and the possibility to have one or no intersection point), as well as to visualise the area between the two curves which will be representing the net excess profit asked in the exercise. The example is fully accessible through the following QR code:

QR code for the example of net excess profit calculation.

The solution of the exercise can be followed in parallel with the tool for demonstration already accessed through their mobile devices as shown above.

a. The rate of profitability of the second investment exceeds that of the first until

$$P_1'(t) = P_2'(t)$$
$$50 + t^2 = 200 + 5t$$
$$t^2 - 5t - 150 = 0 \qquad \text{subtract } 200 + 5t \text{ from both sides}$$
$$(t - 15)(t + 10) = 0 \qquad \text{factor}$$
$$t = 15, -10 \qquad \text{since } uv = 0 \text{ if and only if } u = 0 \text{ or } v = 0$$
$$t = 15 \text{ years} \qquad \text{reject the negative time } t = -10$$

b. The excess profit of plan 2 over plan 1 is $E(t) = P_2(t) - P_1(t)$, and the net excess profit NE over the time period $0 \le t \le 15$ determined in part (a) is given by the definite integral

$$\text{NE} = E(15) - E(0) = \int_0^{15} E'(t)\,dt \qquad \text{fundamental theorem of calculus}$$
$$= \int_0^{15} [P_2'(t) - P_1'(t)]\,dt \qquad \text{since } E(t) = P_2(t) - P_1(t)$$
$$= \int_0^{15} [(200 + 5t) - (50 + t^2)]\,dt$$
$$= \int_0^{15} [150 + 5t - t^2]\,dt \qquad \text{combine terms}$$
$$= \left[150t + 5\left(\frac{1}{2}t^2\right) - \left(\frac{1}{3}t^3\right)\right]\Bigg|_0^{15}$$
$$= \left[150(15) + \frac{5}{2}(15)^2 - \frac{1}{3}(15)^3\right] - \left[150(0) + \frac{5}{2}(0)^2 - \frac{1}{3}(0)^3\right]$$
$$= 1{,}687.50 \text{ hundred dollars}$$

Thus, the net excess profit is \$168,750.

The graphs of the rate of profitability functions $P_1'(t)$ and $P_2'(t)$ are shown in Figure 5.14. The net excess profit

$$\text{NE} = \int_0^{15} [P_2'(t) - P_1'(t)]\,dt$$

can be interpreted as the area of the (shaded) region between the rate of profitability curves over the interval $0 \le t \le 15$.

In the following we show two additional examples of usage WM notebooks for supporting learning concepts from Integration domain:

int [200+5t-(50+t^2), {t, 0, 15}]

Riemann sums (2 of 2)

Assuming "int" is an integral | Use as a math function instead

Definite integrals: Hide steps

```
Integrate[150+5*t-t^2,{t, 0, 15}]
```

$\int_0^{15} (200+5t-(50+t^2))\, dt = \frac{3375}{2}$

Possible intermediate steps:

Compute the definite integral:

$\int_0^{15} (-t^2+5t+150)\, dt$

Integrate the sum term by term and factor out constants:

$= -\int_0^{15} t^2\, dt + 5\int_0^{15} t\, dt + 150\int_0^{15} 1\, dt$

Apply the fundamental theorem of calculus.

The antiderivative of t^2 is $\frac{t^3}{3}$:

$= \left(-\frac{t^3}{3}\right)\Big|_0^{15} + 5\int_0^{15} t\, dt + 150\int_0^{15} 1\, dt$

Evaluate the antiderivative at the limits and subtract.:

$\left(-\frac{t^3}{3}\right)\Big|_0^{15} = \left(-\frac{15^3}{3}\right) - \left(-\frac{0^3}{3}\right) = -1125$

$= -1125 + 5\int_0^{15} t\, dt + 150\int_0^{15} 1\, dt$

Apply the fundamental theorem of calculus.

The antiderivative of t is $\frac{t^2}{2}$:

$= -1125 + \frac{5t^2}{2}\Big|_0^{15} + 150\int_0^{15} 1\, dt$

Evaluate the antiderivative at the limits and subtract.:

$\frac{5t^2}{2}\Big|_0^{15} = \frac{5 \cdot 15^2}{2} - \frac{5 \cdot 0^2}{2} = \frac{1125}{2}$

$= -\frac{1125}{2} + 150\int_0^{15} 1\, dt$

Apply the fundamental theorem of calculus.

The antiderivative of 1 is t:

$= -\frac{1125}{2} + 150t\Big|_0^{15}$

Evaluate the antiderivative at the limits and subtract.:

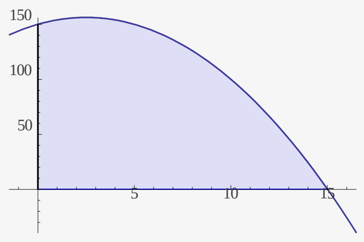
$150t\Big|_0^{15} = 150 \cdot 15 - 150 \cdot 0 = 2250$

Answer:

$= \frac{3375}{2}$

Visual representation of the integral:

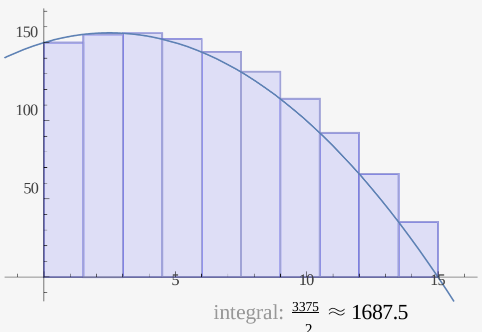
```
Plot[150+5*t-t^2,{t, 0, 15}]
```



Riemann sums: More cases

left sum	$\frac{1125\,(3n^2+2n-1)}{2n^2} = \frac{3375}{2} + \frac{1125}{n} + O\left(\left(\frac{1}{n}\right)^2\right)$

(assuming subintervals of equal length)



integral: $\frac{3375}{2} \approx 1687.5$

Riemann sum: 1794.38

error: 106.875

number of subintervals

summation method: left endpoint | midpoint | right endpoint

Indefinite integrals: Hide steps

```
Integrate[150+5*t-t^2, t]
```

$\int (200+5t-(50+t^2))\, dt = -\frac{t^3}{3} + \frac{5t^2}{2} + 150t + \text{constant}$

Possible intermediate steps:

Take the integral:

$\int (-t^2+5t+150)\, dt$

Integrate the sum term by term and factor out constants:

$= -\int t^2\, dt + 5\int t\, dt + 150\int 1\, dt$

The integral of t^2 is $\frac{t^3}{3}$:

$= -\frac{t^3}{3} + 5\int t\, dt + 150\int 1\, dt$

The integral of t is $\frac{t^2}{2}$:

$= \frac{5t^2}{2} - \frac{t^3}{3} + 150\int 1\, dt$

The integral of 1 is t:

Answer:

$= -\frac{t^3}{3} + \frac{5t^2}{2} + 150t + \text{constant}$

int [5000-20t^2-(2000+10t^2), {t, 0, 10}] »

```
Integrate[5000-20* t^2- (2000+10* t^2),{ t, 0, 10}]
```

Assuming "int" is an integral | Use as a math function instead

Definite integrals:

Hide steps

```
Integrate[3000-30*t^2,{t, 0, 10}]
```

$\int_0^{10} (5000 - 20t^2 - (2000 + 10t^2)) \, dt = 20\,000$

Possible intermediate steps:

Compute the definite integral:

$\int_0^{10} (3000 - 30t^2) \, dt$

Integrate the sum term by term and factor out constants:

$= -30 \int_0^{10} t^2 \, dt + 3000 \int_0^{10} 1 \, dt$

Apply the fundamental theorem of calculus.

The antiderivative of t^2 is $\frac{t^3}{3}$:

$= (-10t^3)\Big|_0^{10} + 3000 \int_0^{10} 1 \, dt$

Evaluate the antiderivative at the limits and subtract.:

$(-10t^3)\Big|_0^{10} = (-10\, 10^3) - (-10\, 0^3) = -10\,000$

$= -10\,000 + 3000 \int_0^{10} 1 \, dt$

Apply the fundamental theorem of calculus.

The antiderivative of 1 is t:

$= -10\,000 + 3000t \Big|_0^{10}$

Evaluate the antiderivative at the limits and subtract.:

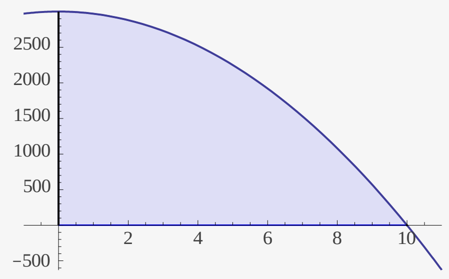
$3000t \Big|_0^{10} = 3000\, 10 - 3000\, 0 = 30\,000$

Answer:

$= 20\,000$

Visual representation of the integral:

```
Plot[3000-30*t^2,{t, 0, 10}]
```

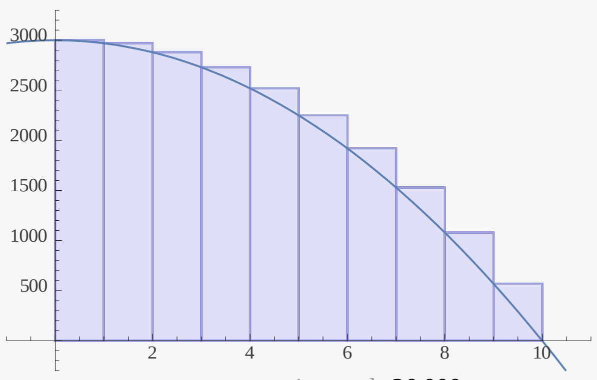


Riemann sums:

More cases

left sum	$\frac{5000\,(4n^2+3n-1)}{n^2} = 20\,000 + \frac{15\,000}{n} + O\left(\left(\frac{1}{n}\right)^2\right)$

(assuming subintervals of equal length)



integral: 20 000

Riemann sum: 21 450.

error: 1450.

number of subintervals

summation method: left endpoint | midpoint | right endpoint

Indefinite integrals:

Hide steps

```
Integrate[3000-30*t^2, t]
```

$\int (5000 - 20t^2 - (2000 + 10t^2)) \, dt = 3000t - 10t^3 + \text{constant}$

Possible intermediate steps:

Take the integral:

$\int (3000 - 30t^2) \, dt$

Integrate the sum term by term and factor out constants:

$= -30 \int t^2 \, dt + 3000 \int 1 \, dt$

The integral of t^2 is $\frac{t^3}{3}$:

$= -10t^3 + 3000 \int 1 \, dt$

The integral of 1 is t:

Answer:

$= 3000t - 10t^3 + \text{constant}$

STEM education: Summer course: teachers4STEM

Selma Özçağ-Hacer İlhan- Emin Özçağ

Contents

1. INTRODUCTION

The subject of learning and teaching mathematics has been one of the subjects that not only mathematicians but also all scientists and even social scientists have been thinking about for many years.

The classical approach in our teaching ways creates passive students and to be improved the situation with mathematics and to be increased the interest for studying it, the classical approach must be changed. The most important goal of the mathematics is to develop skills to the students for understanding the abstract mathematical concepts and solving the reallife problems. Teaching mathematics should be done in a way that students are not concerned with the exact mathematical procedures, but more concerned with learning mathematics as a way of thinking and knowing. Teaching mathematics should be done in such a way as to promote the likelihood that students will be able to transfer what they have learned, to other areas.

The integration of Science, Technology, Engineering and Mathematics, known as STEM education, is a growing area in developed and developing countries (United Nations Educational, Scientific and Cultural Organization [UNESCO], 2010).

One study that looked at STEM education challenges in a variety of nations reported:

Basic mathematics education is still too often boring because: it is designed as formal teaching, centred on learning techniques and memorizing rules, whose rationale is not evident to pupils; pupils do not know which needs are met in the mathematics topics introduced or how they are linked to the concepts familiar to them; links to the real world are weak, generally too artificial to be convincing and applications are stereotypical; there are few experimental and modelling activities, technology is quite rarely used in a relevant manner; pupils have little autonomy in their mathematical work and often merely reproduce activities (UNESCO)

In order to improve STEM education, we need first to improve the learning environment.

More specifically, the question is:

• What technology-enhanced methods could be used to innovate and improve STEM education?

2. Learning environments

What is a Learning Environment?

A learning environment is more than just a classroom—it's **a space in which students feel safe and supported in their pursuit of knowledge, as well as inspired by their surroundings**.

2.1. Teacher-centered learning environments

Most teachers use a variation of a teacher-centered model of instruction, where the emphasis is upon the presentation of a body of knowledge or a set of skills that students are to learn (Goodlad, 1984; Knapp & Glenn, 1996).

In traditional, teaching-based classrooms, activity is the teacher's domain. In this environment students are passive listeners and follows the instructions carefully. Technology in this situation can be a 'patient, non-threatening tutor for basic skill acquisition ... offering students infinite opportunity to repeat problems until process or content is mastered' (Dwyer, 1996, p. 18).

2.2 Student-centered learning environments

Some teachers de-emphasize the teaching process in favor of the learning process. In these classrooms, teachers use a discovery approach to learning where the focus is on the process, and the responsibility for learning is shared with the student. Collaborative, often team-based experiences in these classrooms help students to become deeply involved in manipulating information and thinking about it through processes of inquiry, critical thinking, problem solving, discussion and communication. (Gibson 2001, p 42)

Dwyer (1996) simplifies the differences between teacher-centered (knowledge instruction) and student-centered (knowledge construction) learning environments by charting the attributes of each environment as shown in Table I.

Knowledge Instruction	Knowledge Construction	
Classroom Activity	Teacher-centered (didactic)	Learner-centered (interactive)
Teacher role	Fact teller (always expert)	Collaborator (sometimes learner)
Student role	Listener (always learner)	Collaborator (sometimes expert)
Instructional emphasis	Facts (memorization)	Relationships (inquiry and invention)
Concept of knowledge	Accumulation of facts	Transformation of facts
Demonstration of success	Quantity	Quality of understanding
Assessment	Norm-referenced (multiple-choice items)	Criterion-referenced (portfolios/performances)
Technology use	Drill-and-practice	Communication (Collaboration, information access, expression)

Table I. Attributes of instruction and construction learning environments (adapted from Dwyer, 1996).

In summarizing Dwyer's conception of the role of technology in each of these settings, it appears to be useful to consider the computer as a tutor in the knowledge instruction setting, andas a tool in the knowledge construction setting.

3. STEM EDUCATION

STEM education is a learning environment where students are at the center of learning in the fields of science, technology, engineering and mathematics. This environment has been created with the contributions of schools, teachers and even families, focusing on the social, emotional, physical and scientific needs of the student.

The acronym STEM was initiative first coined by the National Science Foundation (NSF) nearly three decades ago, in 1990s, [1]. Sometimes the term can refer to any-or-all the fields of Science, Mathematics, Engineering and Technology both, individually and in integrated form.

The term "STEM education" refers to teaching and learning in the fields of Science, Technology, Engineering, and Mathematics; typically including educational activities across all grade levels, from pre-school to post-doctorate, and in both formal and informal classroom settings (Gonzalez & Kuenzi, 2012).

What separates STEM from the traditional science and math education is the blended learning environment and showing students how the scientific method can be applied to everyday life. It teaches students computational thinking and focuses on the real-world applications of problem solving. A curriculum that is STEM-based is necessary to include real-life situations to help the students learn.

Technology integration in the curricula entails the teachers and students seamless use of technology as a tool to accomplish a given task in a disciplined student that promotes higher-order thinking skills.

Is technology a part of STEM?

Our question is how to improve STEM education both in high schools and universities ? We will discuss how new forms of learning, including project-based learning and video-game-based learning can significantly boost student STEM learning.

There are many studies on the importance of using educational technologies and developing new technologies to develop and improve STEM learning outcomes.

Educational technologies such as online interactive learning environments, simulation, augmented reality (AR), virtual reality (VR), and digital gaming have a great role in STEM lessons

3.1. Online Interactive Learning

Online interactive learning tools can help to pave the way for active learning allowing students to share valuable information, extract key ideas from new material, and organize a mental framework. These collaborative tools also align with STEM education, which focuses on addressing real problems, intellectual risk-taking and trial-and-error problem-solving, collaboration, and intrinsic motivation. These interactive tools allow teachers to partner with students in the learning process, which is critical for problem-based and student-centered learning.

Google for Education shareable devices and collaborative tools help teachers prepare students with new, more engaging ways to learn the information and skills they need to succeed.

3.2. Simulation

Simulation tools support STEM learning by providing opportunities to manipulate both virtual and actual environments.
Today, simulation is used extensively in classroom settings to train doctors, civil and military, and armed forces personnel for critical, unusual life-threatening conditions and unusual circumstances.

3.3 Augmented Reality and Virtual Reality

The future of learning and workplace training is connected to immersive learning technology, i.e. augmented and virtual reality.
Augmented reality, or AR, is a type of software used on a smart device, such as a tablet, smart eyeglass or smartphone to project digital items, such as a moving cartoon drawing, onto the real image produced by the camera. Virtual reality, or VR, takes

this process a set further. Instead of projecting onto a real environment, VR creates an entirely new digital environment that can be viewed in 360 degrees (Cariker 2018).

3.4 Gaming

Gaming, as an instructional tool, enables educators to create participatory learning activities, assess understanding of complex and ill-formed situations, facilitate critical thinking and problem-solving capabilities, and ensure active engagement across the learning continuum for all students (Raupp 2018)

Project description

Most of the universities in teaching mathematics to STEM students still rely on the traditional nineteen-century approach based on blackboard and chalk. Videos and animation are poorly involved in teaching methodology. Therefore, the main objectives of this project are the design of new interactive methods and practices for teaching mathematics in STEM context in the spirit of active and collaborative learning; implementation of these methods in selected real-life courses, their improvement, and dissemination. These methods will both, attract students to STEM and contribute to the development of highly skilled STEM professionals. Additionally, in the past decades, we are witnessing the rapid development of new technologies that can be exploited in the educational process

4. Exemplary course: Introduction to Game Theory

One of the elective courses we give in the mathematics department of Hacettepe University is the "introduction to game theory ".

Game Theory (Binmore, 1994) provides useful mathematical tools to understand the possible strategies that individuals may follow when competing or collaborating in games. This branch of applied mathematics is used nowadays in the social sciences (mainly economics), biology, engineering, political science, international relations, computer science and philosophy. Initially it was developed to analyze competitions in which one individual does better at another's expense: zero sum games (Morgenstern and von Neumann, 1947)

Generally, in the first lesson, rather than the theoretical definition of game theory I usually talk about the historical development of the game theory, about which science fields it is related to, how mathematics is used in the game theory etc. Then, the example of the prisoner dilemma, which is one of the most popular examples of game theory, is introduced and payoff matrices are created and a brief introduction to Nash equilibrium is given.

This is how the first couple of lessons of a classic game theory course would start before this Project.

Of course we should keep in mind that "Game theory starts with an unfair advantage over most other scientific subjects—it is applicable to numerous interesting and thought-provoking aspects of decisionmaking in economics, business, politics, social interactions, and indeed to much of everyday life, making it automatically appealing to students. " (Dixit, 2005)

Nash Equilibrium

Definition. A strategic form game is composed of
➡ Set of players : N
➡ A set of actions : A_i for each player i
➡ A payoff function: $u_i : A \to R$ for each player i
In general, we name the players by integers and denote a generic player by i, whom we call player i.

We interpret A_i as the set of all available actions (or strategies) to player i. That is, for player i, "playing the game" means choosing an action from the set A_i.

Basic Terminology

Given the players, strategies and payoffs, one can represent the game in normal (or strategic) form, which tells us the associated payoff for every possible combination of strategies. We call each combination of strategies played by the players a strategy profile and we use a matrix to represent the payoff associated with each strategy profile. For example, this matrix:

		Player 2	
		A	B
Player 1	A	2,2	0,1
	B	1,0	1,1

should be interpreted to mean:

- given the profile (A, A), both Players 1 and 2 gain 2 each.
- given the profile (A, B), Player 1 gains nothing, Player 2 gains 1.
- given the profile (B, A), Player 1 gains 1, Player 2 gains nothing.
- given the profile (B, B), both Players 1 and 2 gain 1 each.

We can classify games as **pure**, where players make their choices deterministically, or as **mixed**, where players are allowed to randomize their strategies.

Pure Nash Equilibrium

One of the most important concepts of game theory is the idea of a Nash equilibrium. Consider two players Alice and Bob, who are playing a pure strategy game. We say that Alice and Bob's choice of strategies (the strategy profile) is in Nash equilibrium if

- given Bob's strategy, Alice is playing the best strategy she can (to maximize her payoff), and
- given Alice's strategy, Bob is playing the best strategy he can.
- This concept is important because this strategy pair can be considered stable as neither player has an incentive to deviate from his choice.

Extra Point Question:

You can each earn some extra credit on your term paper. You get to choose whether you want 5 points added to your grade, or 10 points. But there's a catch: if more than 10% of the class selects 10 points, then no one gets any points.

All selections are anonymous, There are about 60 students in class, and all of them vote with googlesheets. (automated response systems)

A Beautiful Mind, which was based, however loosely, on the life of John Nash himself. The crucial scene from the movie, where Nash is supposed to have discovered his concept of equilibrium, shows him in a bar with three male friends.

A blonde and her four brunette friends walk in. All four men would like to win the blonde's favor. However, if they all approach her, each will stand at best a one-fourth chance; actually, the movie seems to suggest that she would reject all four. The men will have to turn to the brunettes, but then the brunettes will reject them also, because "no one likes to be second choice." In the movie, Nash says that the solution is for them all to ignore the blonde and go for the brunettes. One of the other men thinks this is just a ploy on Nash's part to get the others to go for the brunettes so he can be the blonde's sole suitor. If one thinks about the situation using game theory, the Nash character is wrong and the friend is right. The strategy profile where all men go for the brunettes is not a Nash equilibrium: Given the strategies of the others, any one of them gains by deviating and going for the blonde. In fact, Anderson and Engers (2002) show that the game has multiple equilibria, but the only outcome that cannot be a Nash equilibrium is the supposedly brilliant solution found by the Nash character!

Definition: Nash equilibrium of a game G in strategic form is defined as any outcome
$(a^*_1........a^*_n)$ such that

$u_i(a^*_i, a^*_{-i}) > u_i(a_i, a^*_{-i})$ for all $a_i \in A_i$

holds for each player i. The set of all Nash equilibria of G is denoted N(G).

The prisoners' dilemma is a very popular example of a two-person game of strategic interaction, and it's a common introductory example in many game theory textbooks. The logic of the game is simple:

it describes a situation where two prisoners, suspected of burglary, are taken into custody. However, policemen do not have enough evidence to convict them of that crime, only to convict them on the charge of possession of stolen goods.

If none of them confesses (they cooperate with each other), they will both be charged the lesser sentence, a year of prison each. The police will question them on separate interrogation rooms, which means that the two prisoners cannot communicate (hence *imperfect information*). The police will try to convince each prisoner to confess the crime by offering them a "get out of jail free card", while the other prisoner will be sentenced to a ten years term. If both prisoners confess (and therefore they defect), each prisoner will be sentenced to eight years. Both prisoners are offered the same deal and know the consequences of each action (*complete information*) and are completely aware that the other prisoner has been offered the exact same deal (therefore, it's *common knowledge*).

As described before, if both prisoners confess the crime they will be charged an eight years sentence each. If neither confesses, they will be charged one year each. If only one confesses, that prisoner will go free, while the other will be charged a ten years sentence. These can be seen as the respective payoffs for each set of strategies.

		PRISONER 2	
		Confess	Lie
PRISONER 1	Confess	-8 , -8	0 , -10
	Lie	-10 , 0	-1 , -1

Prisoner 1 (P1) has to build a belief about what choice P2 is going to make, in order to choose the best strategy. If P2 confesses he will get either -8 or 0, and if he lies he will get either -10 or -1. It can be easily seen that P2 will choose to confess, since he will be better off. Therefore, P1 must choose the best strategy given that P2 will choose to confess: P1 can either confess ($P1_C$, which pays -8) or lie ($P1_L$, which pays -10). The rational thing to do for P1 is to confess. Proceeding inversely, we analyse the beliefs of P2 about P1's strategies, which gets us to the same point: the rational thing to do for P2 is to confess. Therefore, [$P1_C$, $P2_C$] is the Nash equilibrium in this game (underlined in red).

Mixed strategies

So far we have considered only pure strategies, and players' best responses to deterministic beliefs. Now we will allow mixed or random strategies, as well as best responses to probabilistic beliefs. Many games have no pure strategy Nash equilibrium. But we will discuss why every finite game has at least one mixed strategy Nash equilibrium.

Example: In the battle of the sexes, a couple argues over what to do over the weekend. Both know that they want to spend the weekend together, but they cannot agree over what to do. The man prefers to go watch a boxing match, whereas the woman wants to go shopping. This is a classical example of a coordination game, analysed in *game theory* for its applications in many fields, such as business management or military operations.

Since the couple wants to spend time together, if they go separate ways, they will receive no *utility* (set of payoffs will be 0,0). If they go either shopping or to a boxing match, both will receive some utility from the fact that they're together, but one of them will actually enjoy the activity. The description of this game in *strategic form* is therefore as follows

		q	1-q
		boxing	shopping
p	boxing	(1,3)	(0,0)
1-p	shopping	(0,0)	(3,1)

This game has two pure strategy Nash equilibria, one where both players go to the boxing, and another where both go to the shpping.There is also a mixed strategy Nash equilibrium, in which the players randomize using specific probabilities.

p.q	p.(1-q)
(1-p).q	(1-p).(1-q)

$$\max_p p[q*1+(1-q)*0]+(1-p)[q*0+(1-q)*3]$$

$$\max_p p[q]+(1-p)[(1-q)*3]$$

$$\max_p p[q]+(1-p)[3-3q]$$

$$q-3+3q=0$$

$$4q=3$$

$$q=\frac{3}{4}$$

$$\max_q q[p*3+(1-p)*0]+(1-q)[p*0+(1-p)*1]$$

$$\max_q q[3p]+(1-p)[(1-q)]$$

$$3p-1+p=0$$

$$4p=1$$

$$p=\frac{1}{4}$$

The Mixed startegy Nash equilibria are {(1/4 ,3/4),(3/4 ,1/4)}.

Definition: Let f(x, y) be a function of two variables. The partial derivative ∂f /∂x is the function obtained by differentiating f with respect to x, regarding y as a constant. Similarly, ∂f /∂y is obtained by differentiating f with respect to y, regarding x as a constant.
We often use the alternative notation

$f_x = \partial f / \partial x, \qquad f_y = \partial f / \partial y.$

Example If $f(x, y) = x^2 + xy + y^3 - 1$ then $f_x = 2x + y$, $f_y = x + 3y^2$.

As a limit,

$$\partial f / \partial x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Also, the equation $z = f(x, y)$ defines a surface in 3-dimensional space with x, y, z-axes, and $\partial f / \partial x$ is the gradient of the tangent at a point in the x-direction.

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R+$\LaTeX$=Sweave

Julije Jakšetić, Marjan Praljak, Ana Vukelić

September 15, 2022

Contents

1 R-Introduction and Overview

- Software packages are important tool in modern STEM education
- R is a programming language for statistical computing and graphics
- Open source freeware
- Extensive (online) resources: manuals, examples, forums, packages, ...
- R has a command line interface
- R is easier and more natural to work with when combined with a specialized editor (for example RStudio)

- Datasets can easily be loaded into R from file formats: Excel files, tab delimited files, coma separated files, ...
- Many datasets and real life examples can be found, for example, on websites:

  Kaggle

  `https://www.kaggle.com`

  DASL (the Data and Story Library)
  `https://dasl.datadescription.com/`

  OpenIntro Statistics
  `https://www.openintro.org/`

  Journal of Statistics Education (Data Archive)
  `http://jse.amstat.org/jse_data_archive.htm`

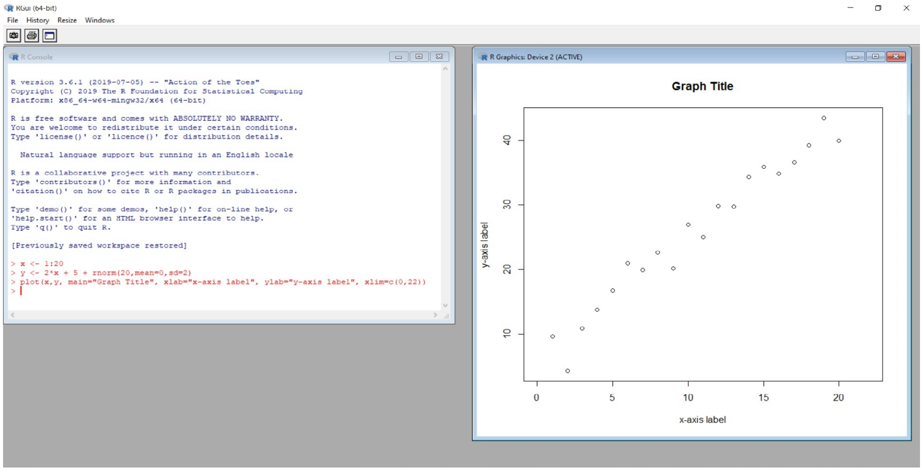


Figure 1: Basic R interface

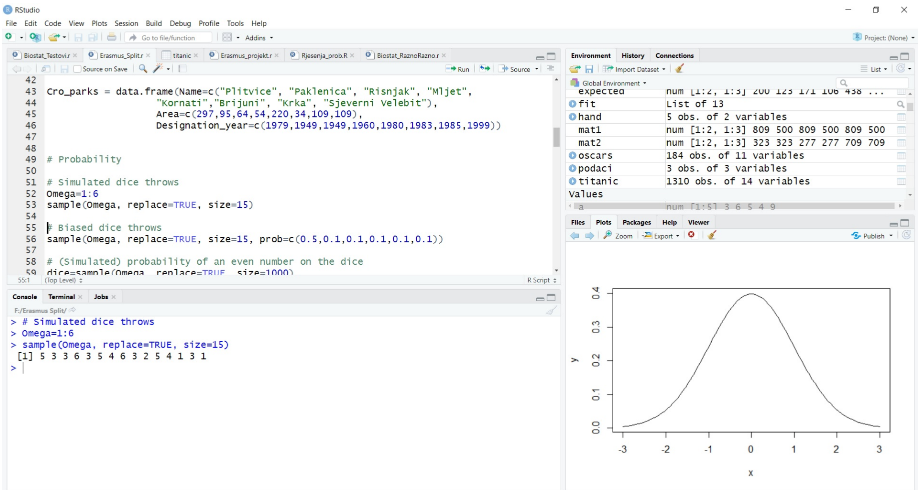


Figure 2: RStudio interface

```
> # Basic R commands
>
> 2+3
```

[1] 5

```
> exp(1)
```

[1] 2.718282

```
> # Vectors in R
>
> a <- c(2,3,1,4,2)
> a
```

[1] 2 3 1 4 2

```
> a[2]
```

[1] 3

```
> a <= 2
```

[1] TRUE FALSE TRUE FALSE TRUE

```
> a[a <= 2]
```

[1] 2 1 2

```
> c(a,a)
```

[1] 2 3 1 4 2 2 3 1 4 2

```
> a+3
```

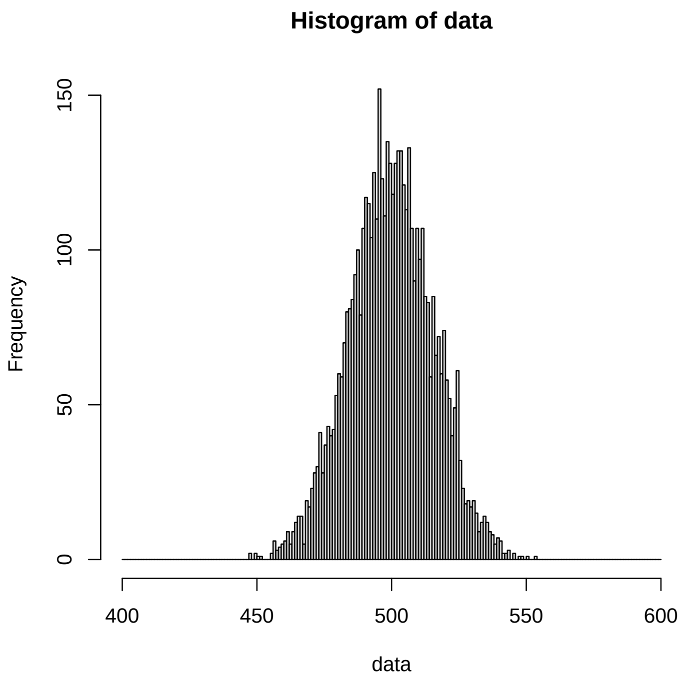
[1] 5 6 4 7 5

```
> a^2
```

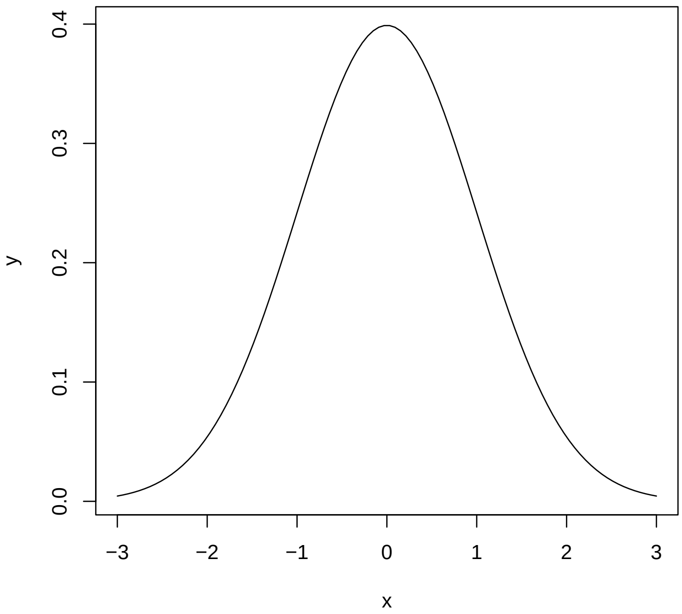
[1] 4 9 1 16 4

```
> # Probability
> # Simulated dice throws
> Omega=1:6
> sample(Omega, replace=TRUE, size=15)

[1] 3 5 4 4 4 3 4 4 6 4 1 4 1 5 3

> # Biased dice throws
> sample(Omega, replace=TRUE, size=15,
prob=c(0.5,0.1,0.1,0.1,0.1,0.1))

[1] 6 1 1 1 3 3 1 3 5 1 5 2 1 6 5

> # (Simulated) probability of an even number on the dice
> dice=sample(Omega, replace=TRUE, size=1000)
> length(dice[dice %in% c(2,4,6)])/1000
```

```
[1] 0.501

> # Binomial distribution approaches normal
> N = 5000;
> data = c();
> for (i in 1:N){
        +   dice=sample(Omega, replace=TRUE, size=1000);
        +   data=append(data,length(dice[dice %in% c(2,4,6)]))
        + }
> hist(data, breaks=400:600)
> # Card decks and card draws
>
> deck =
data.frame(rank=rep(c("A","K","Q","J","T","9","8","7","6","5","4","3","2")
+                   suit=c(rep("Spade",13),rep("Heart",13),
+                          rep("Diamond",13),rep("Club",13)) )
> # Drawing a poker hand (5 cards from a 52 card deck)
> x = sample(1:52,size=5)
> hand = deck[x,]
> hand

rank    suit
49    5    Club
48    6    Club
23    5   Heart
34    7 Diamond
15    K   Heart

> table(hand$suit)

Club Diamond   Heart
2       1       2

> # Probability distributions in R
>
> # Normal distribution in R is denoted by norm
>
> # Simulated sample from normal distribution
> rnorm(n=25, mean=5, sd=2)

[1]  4.44982758  4.64713981  4.56605010  4.78388947 10.48468066
6.73072665
[7]  5.67870128  5.53820923  8.00520022  4.12562472  6.04492477
5.27985588
[13]  3.99612208  0.06965784  3.11899825  5.92281589  7.21052818
4.14751197
[19]  2.90834120  9.82359428  8.10833161  5.60462973  4.87112349
5.75147551
[25]  2.95895057

> hist(rnorm(n=2500, mean=5, sd=2))
```




```
> # Density function of the (standard) normal distribution
>
> x <- seq(-3,3,length=100)
> y <- dnorm(x,mean=0,sd=1)
> plot(x,y,type="l")
```




```
> # Cummulative distribution function
> pnorm(0,mean=0,sd=1)

[1] 0.5

> # Inverse of the CDF, i.e. quantile
> qnorm(0.95,mean=0,sd=1)

[1] 1.644854

>
```

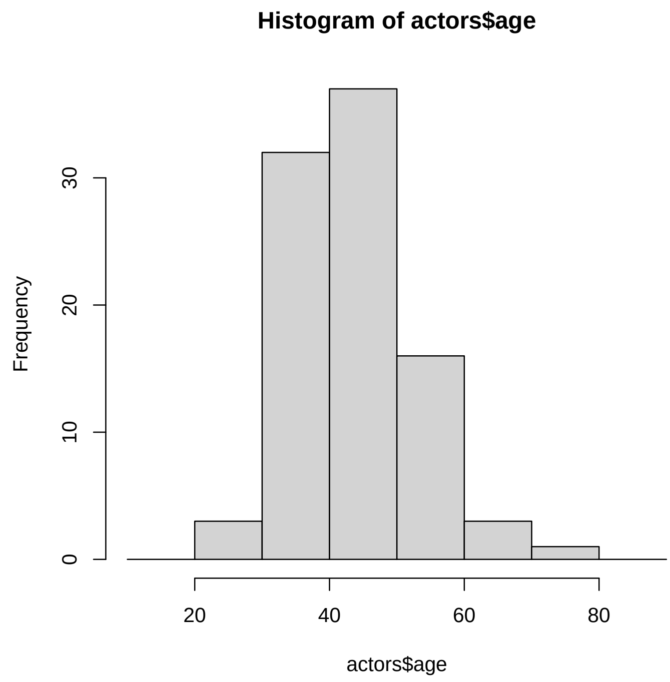
```
> oscars <- read.csv('oscars.csv')
> actresses=subset(oscars,award=="Best actress")
> x <- actresses$age
> mean(x)

[1] 36.25

> sd(x)

[1] 11.90403

> actors=subset(oscars,award=="Best actor")
> y <- actors$age
> mean(y)
```

```
[1] 43.84783
```

```
> sd(y)
```

```
[1] 8.825456
```

```
> hist(actors$age, breaks=seq(10,90,by=10))
```




```
> t.test(x,y)
```

```
Welch Two Sample t-test

data:  x and y
t = -4.9178, df = 167.83, p-value = 2.072e-06
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
-10.647885  -4.547767
sample estimates:
mean of x mean of y
36.25000  43.84783
```

```
> t.test(x,y, var.equal = TRUE)
```

```
Two Sample t-test
```

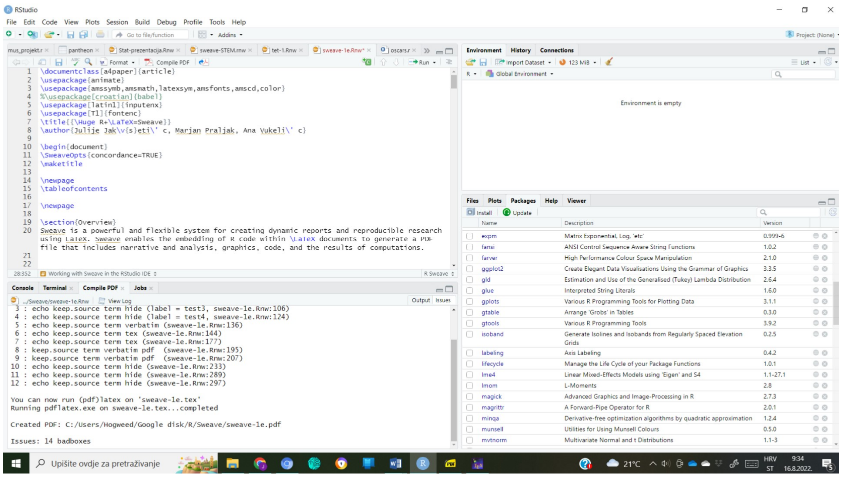
```
data:  x and y
t = -4.9178, df = 182, p-value = 1.948e-06
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
-10.646161  -4.549492
sample estimates:
mean of x mean of y
36.25000  43.84783

> t.test(x,y, var.equal = TRUE, alternative="less")

Two Sample t-test

data:  x and y
t = -4.9178, df = 182, p-value = 9.739e-07
alternative hypothesis: true difference in means is less than 0
95 percent confidence interval:
-Inf -5.043593
sample estimates:
mean of x mean of y
36.25000  43.84783
```

```
> titanic <- read.csv('titanic.csv')
> table(titanic$pclass,titanic$survived)

0   1
1 123 200
2 158 119
3 528 181

> chisq.test(table(titanic$pclass,titanic$survived))

Pearson's Chi-squared test

data:  table(titanic$pclass, titanic$survived)
X-squared = 127.86, df = 2, p-value < 2.2e-16
```

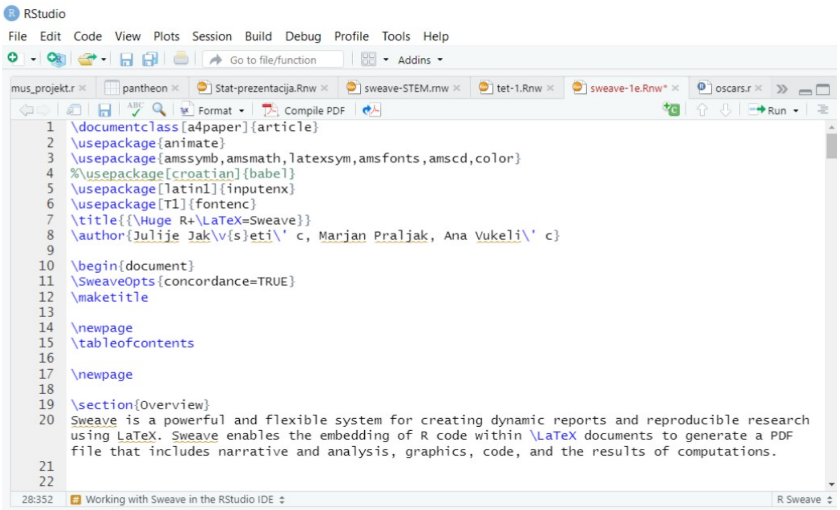
2 Sweave

Sweave is a powerful and flexible system for creating dynamic reports and reproducible research using LaTeX. Sweave enables the embedding of R code within LaTeXdocuments to generate a PDF file that includes narrative and analysis, graphics, code, and the results of computations.

3 Working with Sweave in the RStudio IDE



To start a new Sweave document, go to **File | New and select "R Sweave"**. This will provide a basic Sweave template. From here, you can enter text and LaTeX commands. R chunks can also be inserted to interweave R commands and output into your document. To insert an R chunk, use the **Chunks** menu at the top right of the source editor.



The first thing is set *working directory* inside R–studio: this directory contains your `sweave-stem.rnw` file.
It is mandatory to put `Sweave.sty` file inside this folder.

The scheme process:

$$\texttt{sweave-stem.Rnw} \xrightarrow{\texttt{Sweave}} \texttt{sweave-stem.tex} \xrightarrow{\texttt{pdflatex}} \texttt{sweave-stem.pdf}\text{(Voila!)}$$

We now list options to generate powerful pdf document.

The left hand side are `"chunks"` inside `Sweave`, and right hand side are output form inside pdf document.

No additional options

`<<label=test1>>=`
x=2
`# calculate x+1`
x+1
@
	`> x=2`
	`> # calculate x+1`
	`> x+1`
	
	`[1] 3`

Option *echo*

`<<test2,echo=FALSE>>=`
`# calculate x+2`
x+2
@
	`[1] 4`

Option *results*

`<<test3,results=hide>>=`
`# calculate x+3`
x+3
@
	`> x+3`

Option *keep.source*

`<<test4,results=hide,keep.source=true>>=`
`<<test1>>`
@
	`> x=2`
	`> # calculate x+1`
	`> x+1`

4 Import and export of data

Let us use the file `goals.txt` from the web page:

```
>goals=read.table("https://www.fsb.unizg.hr/usb_frontend/
files/1339628887-0-golovi.txt", header=TRUE , dec=",")
>goals
```

```
> goals= read.delim("golovi.txt")
```

In order to manipulate with the loaded data we have to install the package(inside R) `xtable`

```
> library(xtable)
> tab=xtable(goals)
```

This produces LaTeX code inside document for the table:

	numofgoals	frek
1	0	30
2	1	79
3	2	99
4	3	67
5	4	61
6	5	24
7	6	11
8	7	6
9	8	2
10	9	1
11	10	0

5 Some of symbolic and numeric calculations in R and its LaTeX design

Example For a given function $f(x) = \ln(\ln x)$ find the values of $f(10)$, $f'(10)$ (round to 5 decimal places). Skicirajte graf funkcije f na intervalu $(1, 100)$.

```
> f = function(x) log(log(x))
> y=expression(log(log(x)))
> D(y,"x")
```

1/x/log(x)

```
> f1=function(x) 1/x/log(x)
```

We read, symbolic, derivation of the function f; $f'(x) = 1/x/\log(x)$.

The value of the function f at 10 is $f(10) = 0.83403$ and the value of its derivation is $f'(10) = 0.04343$.

The graph of f

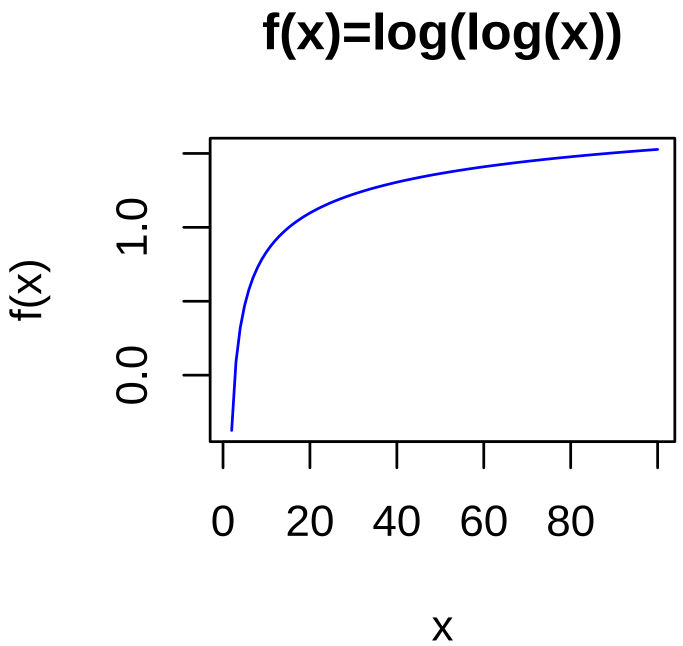


Figure 3: The graph of f

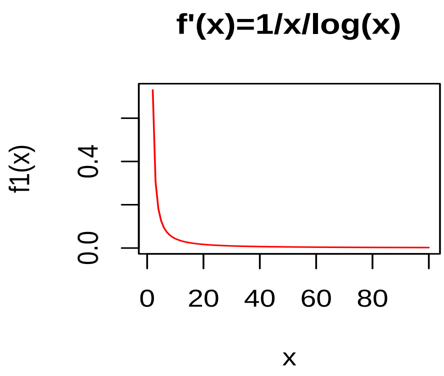


Figure 4: The graph of f'

6 Animations

Example. Make the Taylor approximation of the exponential function up to 5th power.

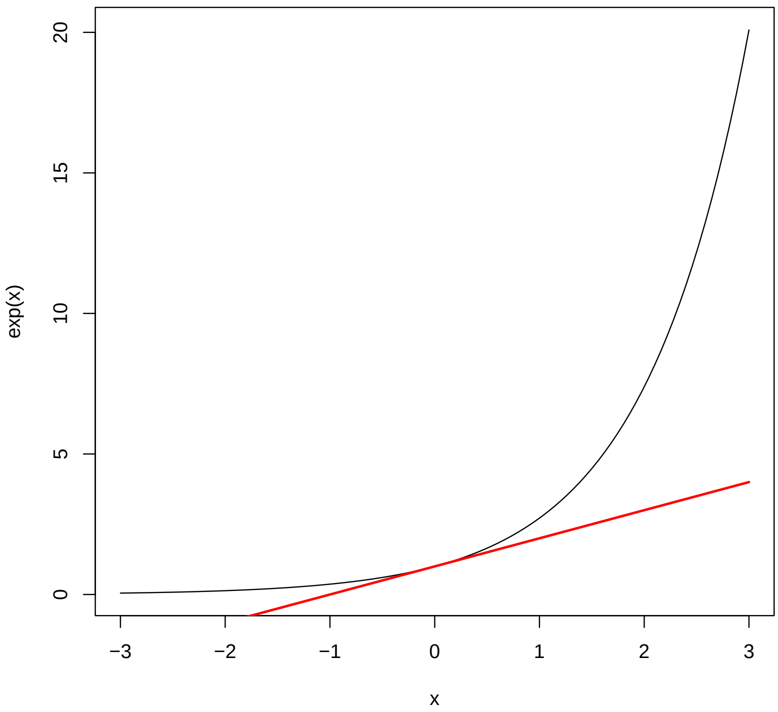
We will visually demonstrate approximation on the interval $(-3, 3)$.

We first generate picture using `R`; the generated picture, we will name it `poco-1b.pdf`

```
> pdf(file="poco-1b.pdf")
> curve(exp(x), from = -3, to = 3)
> curve(1+x, from =  -3, to = 3,add=T,col = "red",lwd=2)
> curve(exp(x), from = -3, to = 3)
> curve(1+x+x^2/2, from = -3, to = 3,add=T,col = "red",lwd=2)
> curve(exp(x), from = -3, to = 3)
> curve(1+x+x^2/2+x^3/6, from =-3, to = 3,add=T,col = "red",lwd=2)
> curve(exp(x), from = -3, to = 3)
> curve(1+x+x^2/2+x^3/6+x^4/24, from = -3, to = 3,add=T,col = "red",lwd=2)
> curve(exp(x), from = -3, to = 3)
> curve(1+x+x^2/2+x^3/6+x^4/24+x^5/120, from = -3, to = 3,add=T,col = "red",lwd=2)
> dev.off()                          # close pdf device
```

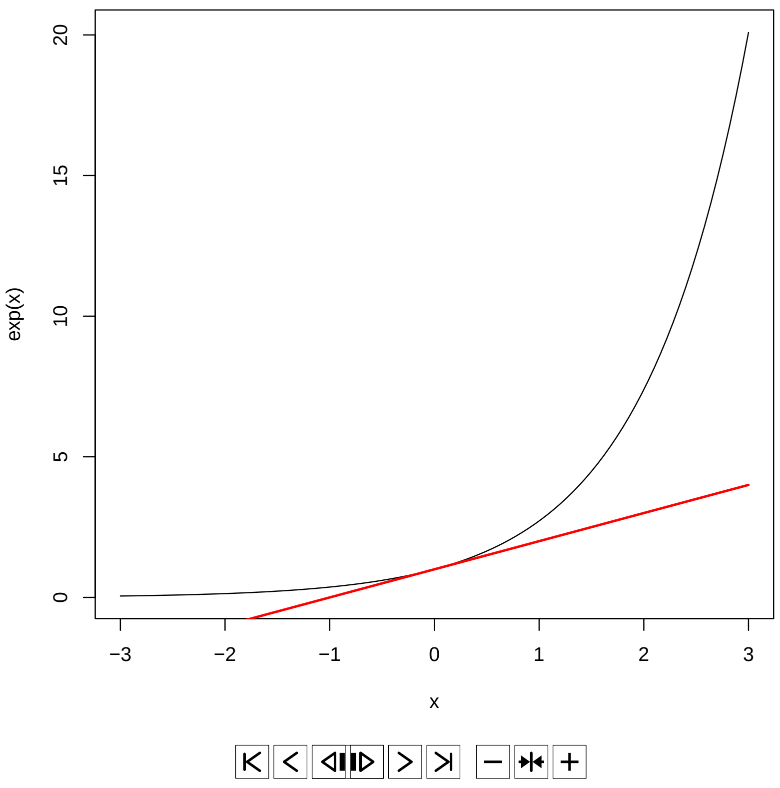
The approximation effect, we get a sense of using LaTeX animation

```
\begin{center}
\animategraphics[loop,autoplay,width=.9\linewidth]{1}{poco-1b}{}{}
\vspace{-5mm} \end{center}
```



We can control the speed of the animation if we add additional command inside LaTeX`controls`

```
\begin{center}

\animategraphics[loop,controls,width=.9\linewidth]{1}{poco-1b}{}{}
\vspace{-5mm}

\end{center}
```



Example . Using animation, give an interpretation of 95% confidence interval.

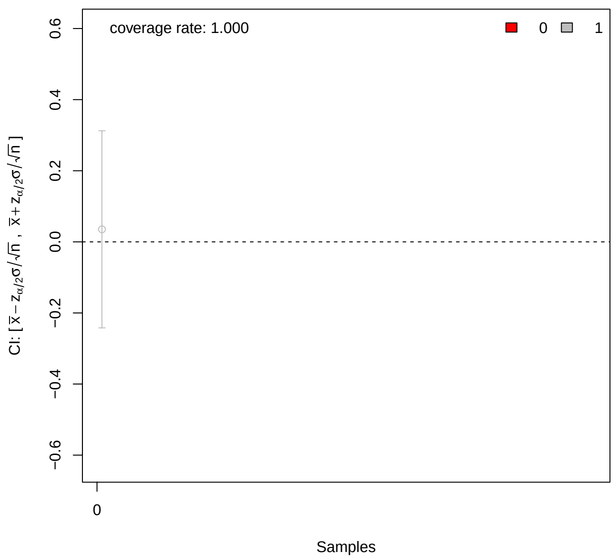
We set a code in R:

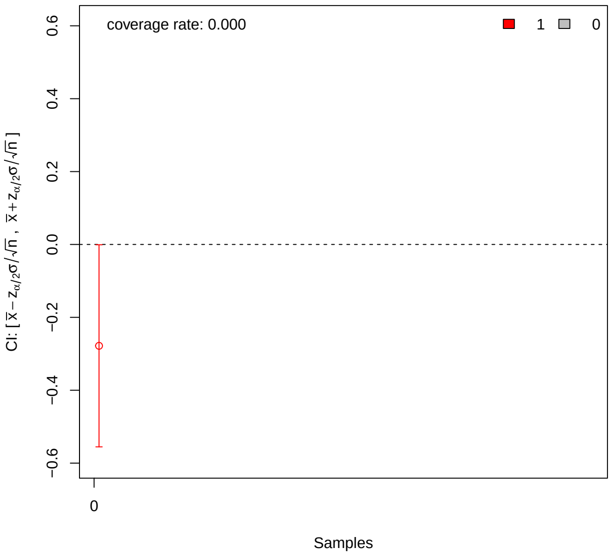
```
> library(animation)
> pdf(file="pac.pdf")            # open pdf device
> ani.options(nmax = 100, interval = 0.25)
> conf.int(level=0.95)
> dev.off()                           # close pdf device
```

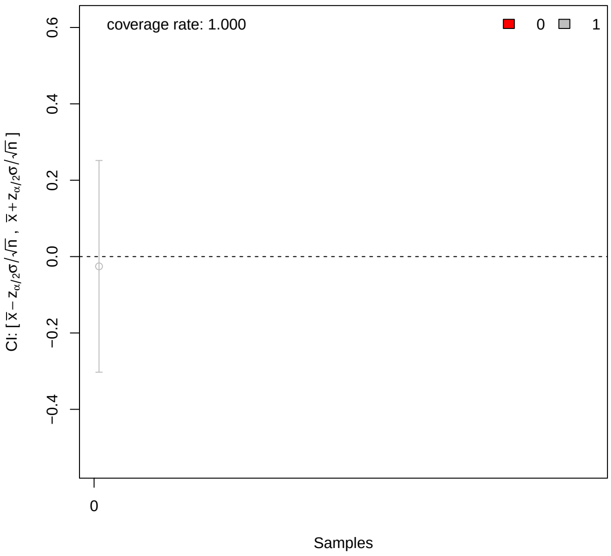
and once again

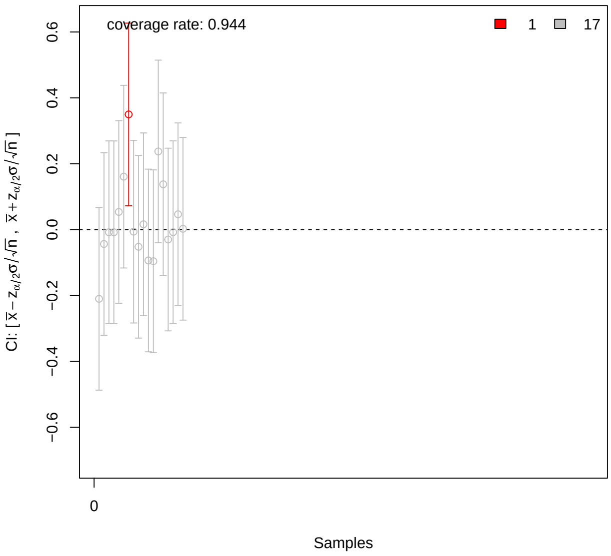
```
> library(animation)
> pdf(file="pac01.pdf")            # open pdf device
> ani.options(nmax = 100, interval = 0.15)
> conf.int(level=0.95)
> dev.off()                           # close pdf device
```

Now we run animation in LaTeX




coverage rate: 0.000
1
0
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
0
Samples


coverage rate: 1.000
0
1
CI: [x̄ − z_{α/2}σ/√n , x̄ + z_{α/2}σ/√n]
0.6
0.4
0.2
0.0
−0.2
−0.4
0
Samples

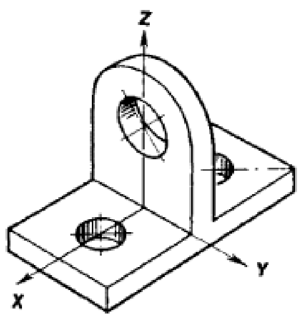

coverage rate: 0.944
1
17
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
CI: [x̄ − z_{α/2}σ/√n , x̄ + z_{α/2}σ/√n]
0
Samples

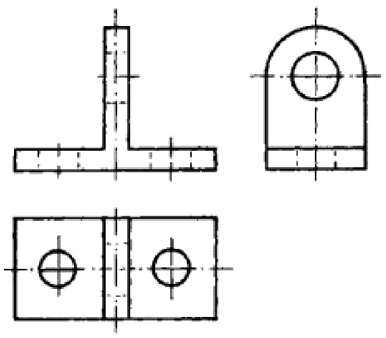
LTT Activity C1: Summer course Teachers4STEM
22nd to 26th August 2022, Split, Croatia

University of Split, Croatia
LECTURE 1: DESCRIPTIVE GEOMETRY IN *STEM* CONTEXT

The challenges in teaching descriptive geometry

Descriptive geometry is a mathematical discipline that investigates 3D objects by means of 2D techniques. The need for the descriptive geometry arose directly from the engineering practice: it is often necessary to present standard objects that engineers (civil and machine engineers, architects, geodesists *etc*.) use, as ideal mathematical forms in order to analyze, measure or interpret them in 3D space. Descriptive geometry thus became unavoidable part of teaching the future engineers. They learn how to visualize objects in 3D space and project them on 2D surface, where they point out their required properties.



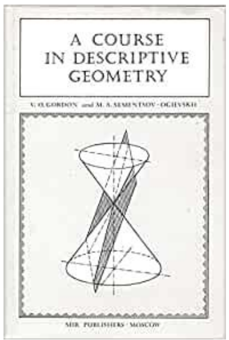


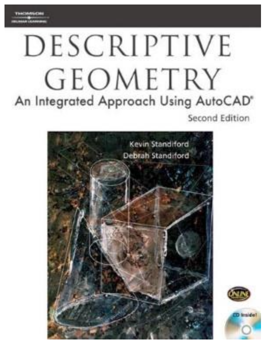
Vice versa, they recognize main properties of 2D projections needed for the reconstruction in 3D space. Thus they develop space visualization skills and specific communication in 3D – 2D ambient achieved in various media: paper (sketch, draft, construction) or monitor: by a fast changing of parameters of projections one can simulate moving an object and see all its sides from different views, all on 2D surface.

Descriptive geometry in Croatia has been taught at four universities in Zagreb, Split, Osijek and Rijeka. The concept is similar, although the programmes differ according to the specific demands of different studies, in the first place civil engineering and architecture, then geodesy and machine engineering.

What certainly mitigated the challenges of teaching descriptive geometry today was founding of the *Croatian Society for Geometry and Graphics* in 1994. This society coordinates the work of teachers, primarily on faculties in Croatia by organizing and realizing various activities with the purpose up-to-date the educational system in teaching descriptive geometry and related topics. Very important scope is cooperation with the similar institutions and associations in other countries, especially by organizing annual conferences or colloquiums of the international character. These encourage and help teachers when dealing with constructive geometry and computer graphics in their professional and scientific improvement. One could say that CSGG actually started the process of STEM education within descriptive geometry in Croatia which branches in various ways today, and one of the opportunities for such orientation is mathSTEM project.

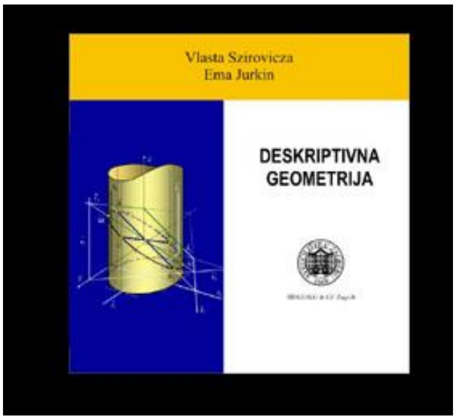
In the beginning, descriptive geometry was taught as any other mathematics discipline, using chalk and blackboard, a pair of drawing triangles and a compass, for both lectures and exercises.





The books that were used were quite often the old editions, since new ones, even the scripts, demanded a complicated process in the lack of appropriate and easy to use programmes.

Modern variants of classical textbooks were better adjusted to the students' needs, but still, not enough to ease the process of teaching descriptive geometry for both students and teachers. One big step further was publishing e-textbook *Descriptive geometry* in 2006 by CSGG.



Since the majority of the teaching materials were covered by this e-textbook, where lessons with constructive steps are made through PowerPoint presentations, it was much easier for students to cope with the lessons that were previously explained on lectures. The teacher suddenly had more time for extra explanations that emerged by this step-by-step constructions on screen, during the basic lectures. Not all details had to be done on the blackboard anymore.

Anyway, exercises, as an important part of teaching descriptive geometry left unchanged: it was up to teacher to find out the way or the programme which would fit best within the process of construction tasks.

In 2020, the pandemics caused by COVID-19 actually hurried this process of the transition to the modern technologies. It coincided with the beginning of the mathSTEM project that we, as descriptive geometry teachers, were also involved. Both lock-down and mathSTEM made us acting vigorously in order to find the appropriate way of teaching, not only lectures, but also exercises with sophisticated constructions via computer aided design, or simply, CAD programs.

Here we focus on two such programs that we used both in the regular teaching process and within mathSTEM lessons that were analysed and recorded. GeoGebra was applied in the process of teaching the Monge method, due to its educational purpose and the possibility of showing variability of solutions on the basic single construction type. Besides, students in Croatia are offered teaching materials www.grad.hr/geometrija/udzbenik that include a variety of examples solved in GeoGebra which makes any new task easier to follow when constructed via the same program. Another program is a more sophisticated one, AutoCAD, which is often used in the engineering practice, especially among architects.

The Monge method (IO3)

Original method of descriptive geometry is named after French mathematician Gaspard Monge (1746-1818) who developed it and is nowadays known as the Monge method. In recognition of his contribution, his name is inscribed among 72 names of French scientists on the base of the Eiffel Tower.

Object are simultaneously orthogonally projected on (at least) two planes of projections or even three, as is shown below.

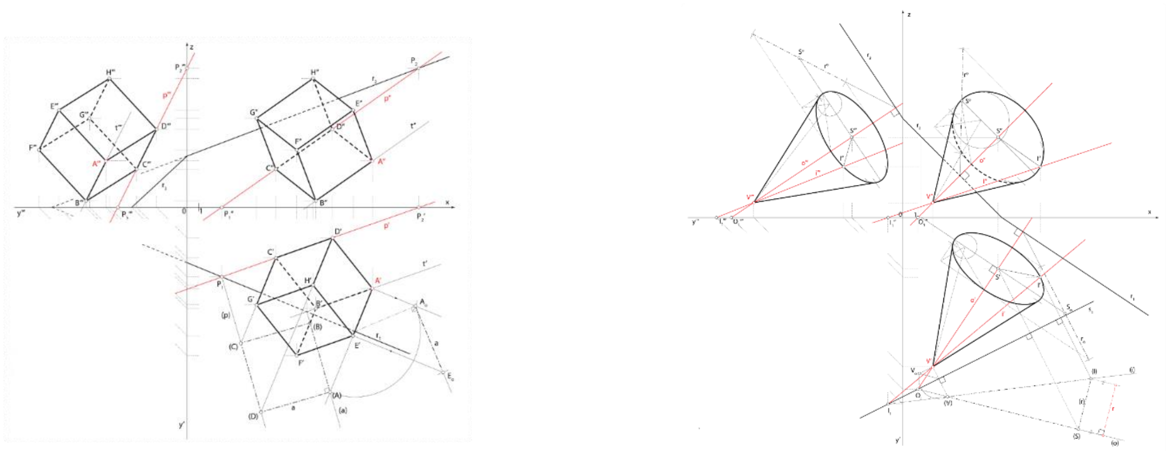
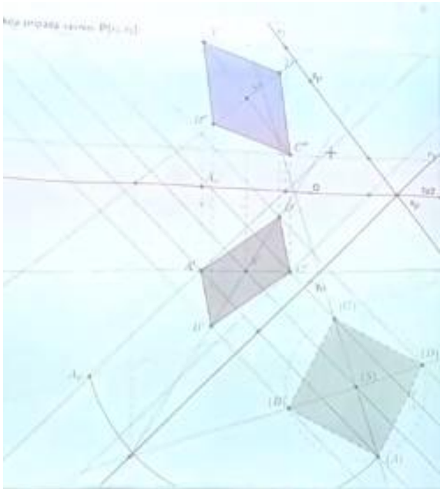


Image source: www.grad.hr/geometrija/udzbenik

In order to construct a solid in the Monge method within IO3 of mathSTEM project, students were first asked to construct the square as the basis of a quadrilateral pyramid. The additional task of upgrading the polygon into a solid was given as a homework.

The construction was done in GeoGebra under the supervision of teachers, in the computer lab. All student were sitting at their own computers following the steps of one of the teachers, on the screen.

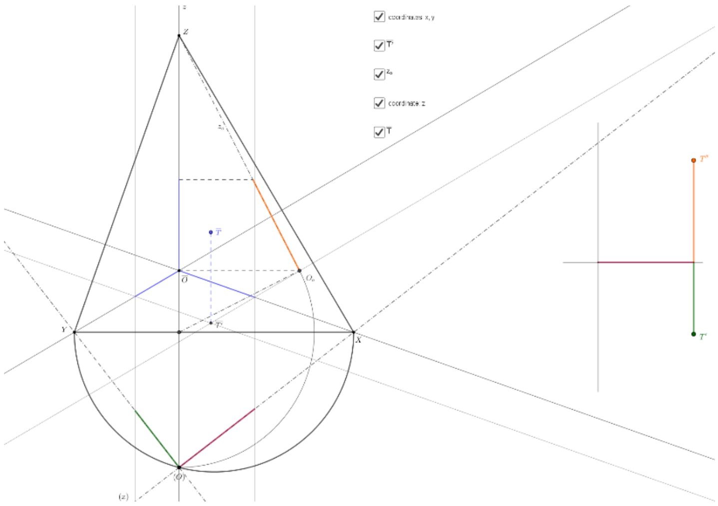


The constructive process made use of previously comprehended laws and techniques. At the end of the task, one of the advantages of GeoGebra was applied: by controlled moving of some parameters, one can see the exchanging sequence of different solutions of the same problem, due to different positioning. Although all students finished the task successfully in required time, the problem that nevertheless occurred later on was that the same students were not able to solve analogous tasks during the regular written exam. It also showed us that students best respond to recorded lectures that imitate chalk-blackboard approach: constructions with the simultaneous explanations of the teacher. Other variants such as recorded teacher's voice that follows PowerPoint presentations or step-by-step constructions *e.g.* in GeoGebra, but without recorded voice interpretations are less effective in this short time learning process.

Domes in orthogonal axonometry: where STEM becomes STEAM (IO3, IO4)

With the Monge method as the starting point, many other methods in descriptive geometry are developed. Axonometric methods *e.g.* present an object together with its coordinate system and by parallel rays project both onto the 2D surface. Thus we get a complete 3D image of an object on a unique plane of projection. Axonometric methods have their specific advances in engineer practice, but also originate from the Monge method and then branch in different sub-methods.

One of those methods is called *orthogonal axonometry*. This one is taught in detail within the course of descriptive geometry for future architects. In the image below, the orthogonal axonometry of a point *T* is shown when constructed step-by-step in GeoGebra in the characteristic trace triangle of the method.



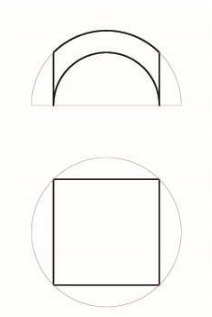
Various domes in architecture can be nicely and rather easily constructed by making use of this method. Contour circle of a dome stays with the same given radius and the horizontal section of the sphere needs a standard axonometric construction.

When we mention types of domes in architecture, we actually enter into a new scope of STEM teaching, which includes recognizing the elements from the history of art in the constructive process of STEM area. This extra "A" that stands for arts in the title is thus justified.

Beside the previously comprehended laws and techniques on orthogonal axonometry, students were also introduced to analysis of domes in architecture.

Three main types of domes in architecture are *spherical, bohemian* and *pendentive* domes. These are simultaneously presented by their pairs of orthogonal projections where the geometric elements of their architectural concept can be properly recognized.

The *spherical dome* is constructed over the square basis of an object:



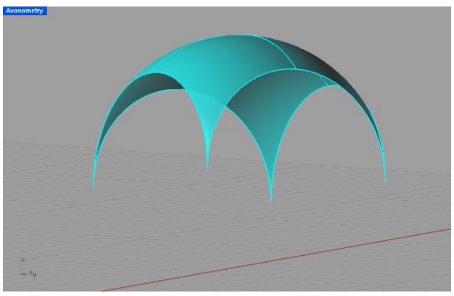


Image source: CSGG

The *pendentive dome* is a spherical dome upgraded by a half-sphere and supported by pendentives (spherical triangles):

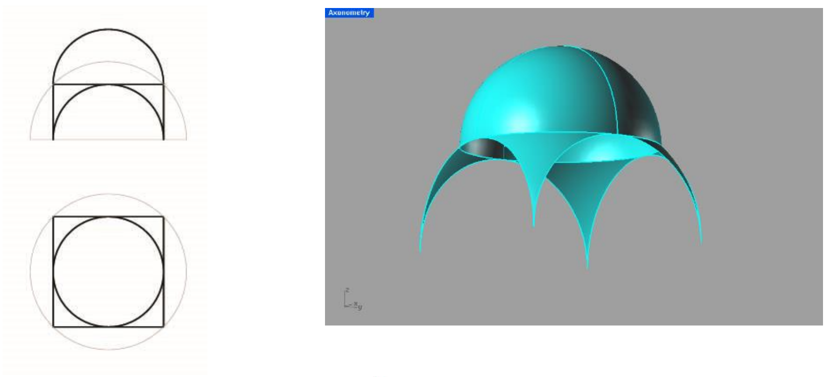


Image source: CSGG

The *bohemian* dome is constructed over the rectangular basis of an object:

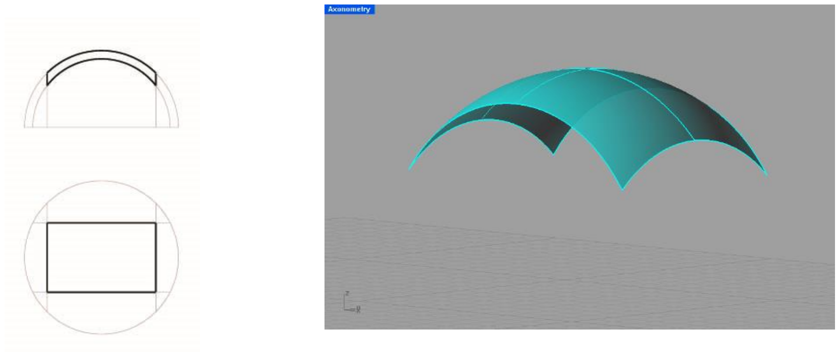
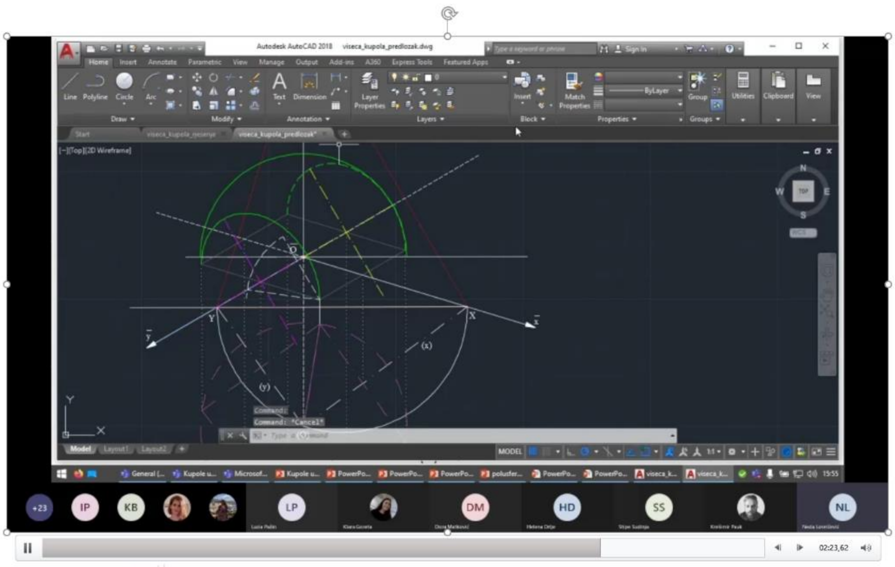
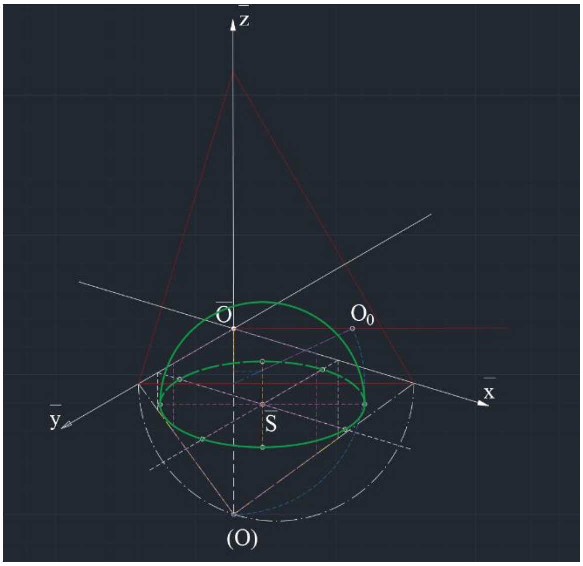


Image source: CSGG

For the constructive process for the lesson recorded within IO3 students were given a template with the trace triangle. This time, due to pandemics, the lesson was held via MS Teams and using AutoCAD. The task was a construction of the *spherical* dome in orthogonal axonometry.



The additional task, as a homework, was upgrading the spherical dome with an extra half-sphere into a *pendentive* dome.



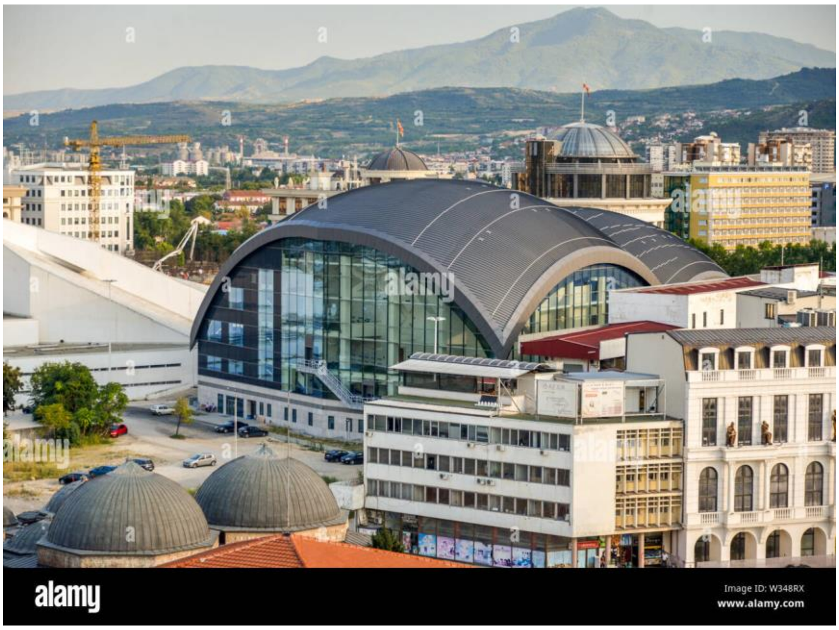
Although GeoGebra suffices for most educational purposes in descriptive geometry, AutoCAD is more powerful tool which architects often use in their everyday practice for it offers a variety of possibilities and is a good choice for more sophisticated constructions, such are the three types of domes.

The element of including arts into the pure constructive process was helpful within IO4 of the mathSTEM project in Ohrid. There we had an extra challenge of explaining the lesson to STEM students who for the most part are not familiar with the concept of descriptive geometry. This approach with the examples from everyday life helped with a motivation.

Finally, learning constructions in any method of descriptive geometry and in any program tool should be always supported by examples from everyday life and according to specific purpose of teaching. In the following images one can see examples in the context of IO4 which was held in N. Macedonia.

Saint Clement of Ohrid, Skopje, N. Macedonia

Monastery of Saint Naum, Ohrid, N. Macedonia



Macedonian Philharmonic, Skopje, N. Macedonia

Conclusion

Although student cope very successfully with various computer programmes, their geometry skills are often left behind in the process. The advantages are students' motivation and skills and new technologies orientation, quick problem solving. The disadvantages are (sometimes) too narrowly oriented learning, not reading or analysing enough. At this moment, it seems that the best approach is a balance between the „old-school" with constant reviewing and questioning and making use of different computer aided designs (CAD programmes). Probably in the near future, when we reach a critical mass of teaching materials completely adjusted to computers, we will be ready for (almost) complete transfer to the new technology. mathSTEM is definitely a „fast forward button" to this expected point.

University of Split, Croatia

LECTURE 2: DESCRIPTIVE GEOMETRY AS A DYNAMIC GEOMETRY

At our Faculty of Civil Engineering, Architecture and Geodesy, we have following courses on four different studies:

Study	Course	Semester	Hours	ECTS
Undergraduate University Study of Civil Engineering	Descriptive Geometry Applied Geometry	I. II.	30+30 30+30	5,0 5,0
Undergraduate University Study of Architecture and Urban Planning	Principles of Projections I Principles of Projections II	I. II.	30+30 30+30	5,0 5,0
Undergraduate University Study of Geodesy and Geoinformatics	Computer Geometry	II.	30+30	5,0
Undergraduate Professional Study of Civil Engineering	Descriptive Geometry	II.	30+30	5,0

Since one of the main purposes of descriptive geometry is how to see and understand 3D relations and their appearance on 2D surface, we list several important learning outcomes for the above courses:

– solve 2D and 3D problems using various methods of projection (Monge's projection; the orthogonal projection, the axonometric projection, the central projection – perspective);
– construct intersections of surfaces and planes;
– construct intersections of two solids;
– analyze topographic maps and draw cuts and fills along a level or grade road and dam;
– solve roof structures of the building;
– construct shades and shadows of various objects.

They all have in common next specific tasks:

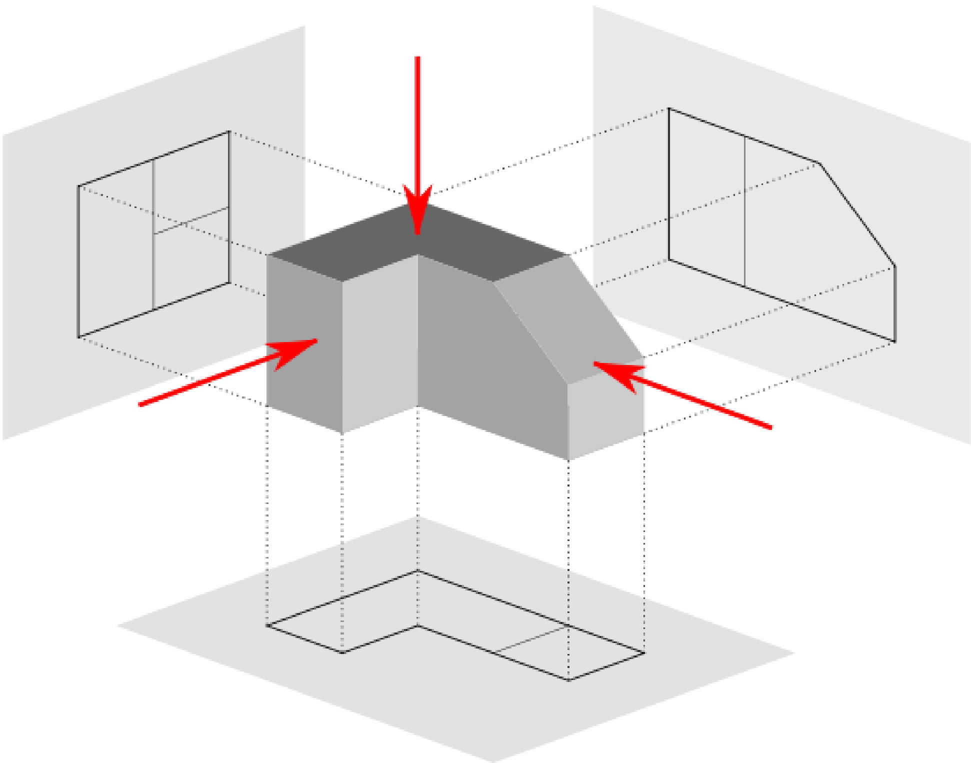
– recognize the laws of projections and apply them accordingly to solve different constructive problems in civil engineering, architecture and geodesy.
– draw and solve constructive tasks using computer programs, CAD software, dynamic geometry software.

Here we present several topics from our courses with examples how to teach and present lectures to students.

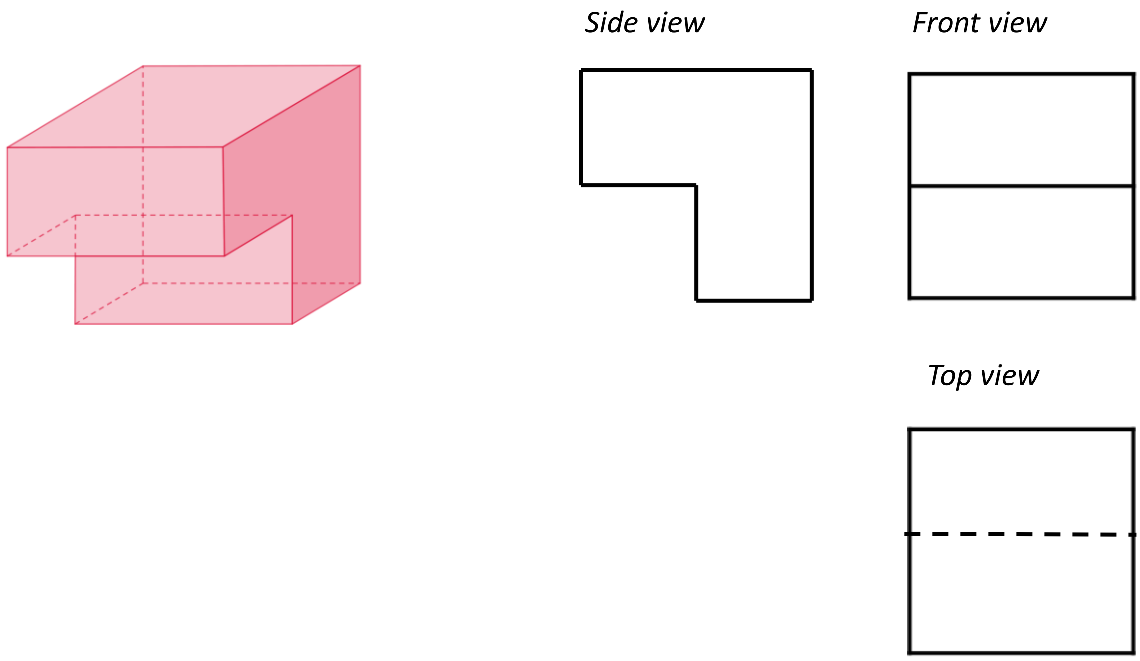
Orthogonal projection: top, front and side view of an object

Each object has its orthogonal top, front and side view, as in the next two examples:

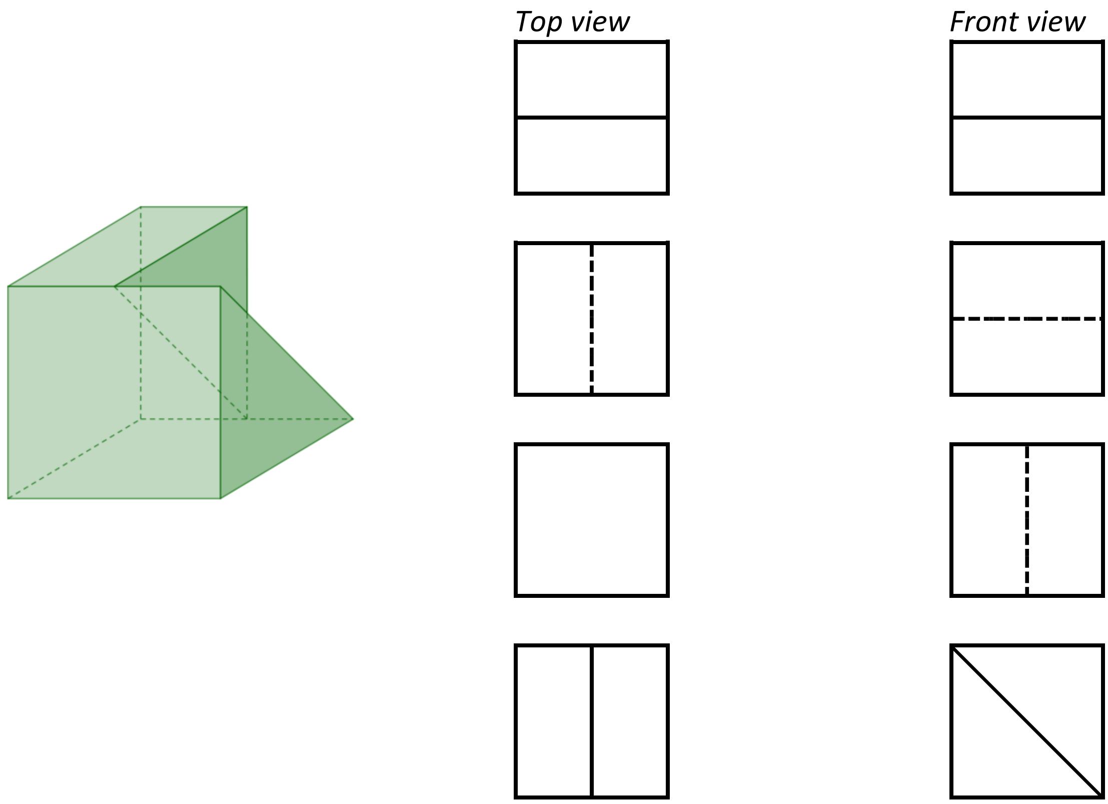
Example 1:



Example 2:



Next we have an example of the student's task: for a given object, choose the correct top and front view obtained by orthogonal projections:

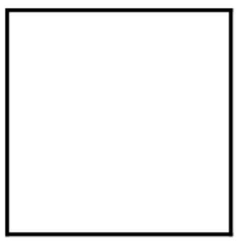


How to obtain and recognize these projections is a first step to get familiar with axonometric projections – projections where the principal axes of an object are *not* orthogonal to the projection plane.

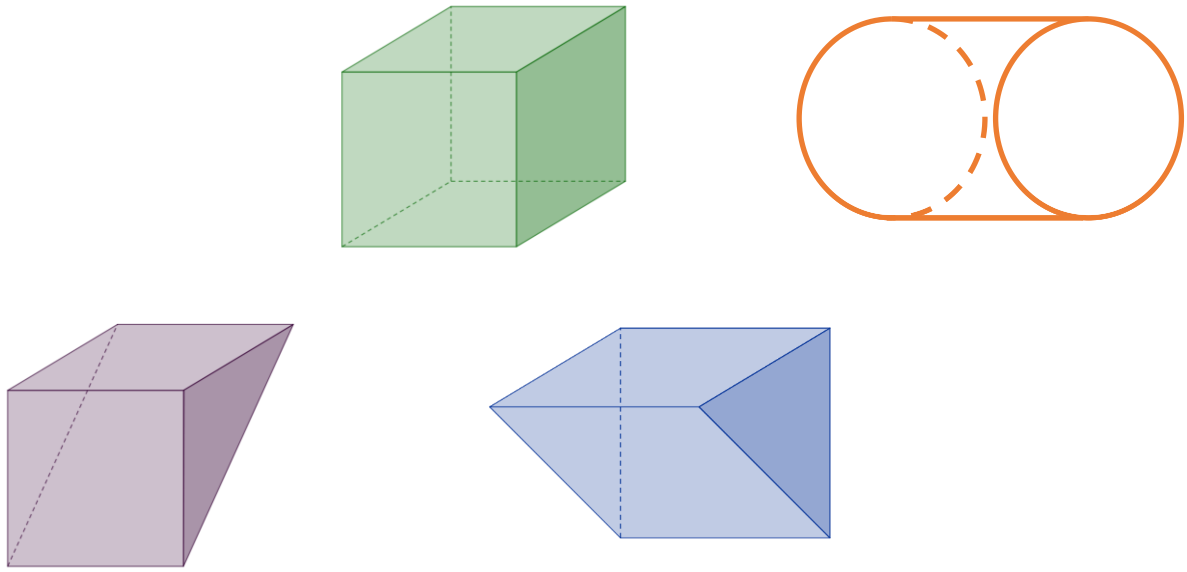
From projections to one or more objects

Often with only two orthogonal projections of an object, one can derive more than one possible 3D shapes that correspond to them. Moreover, some tasks can provide numerous solutions. In this manner, students can significantly develop their space visibility skills. For example, let have next two projections, that are in this case the same:

Front and top view

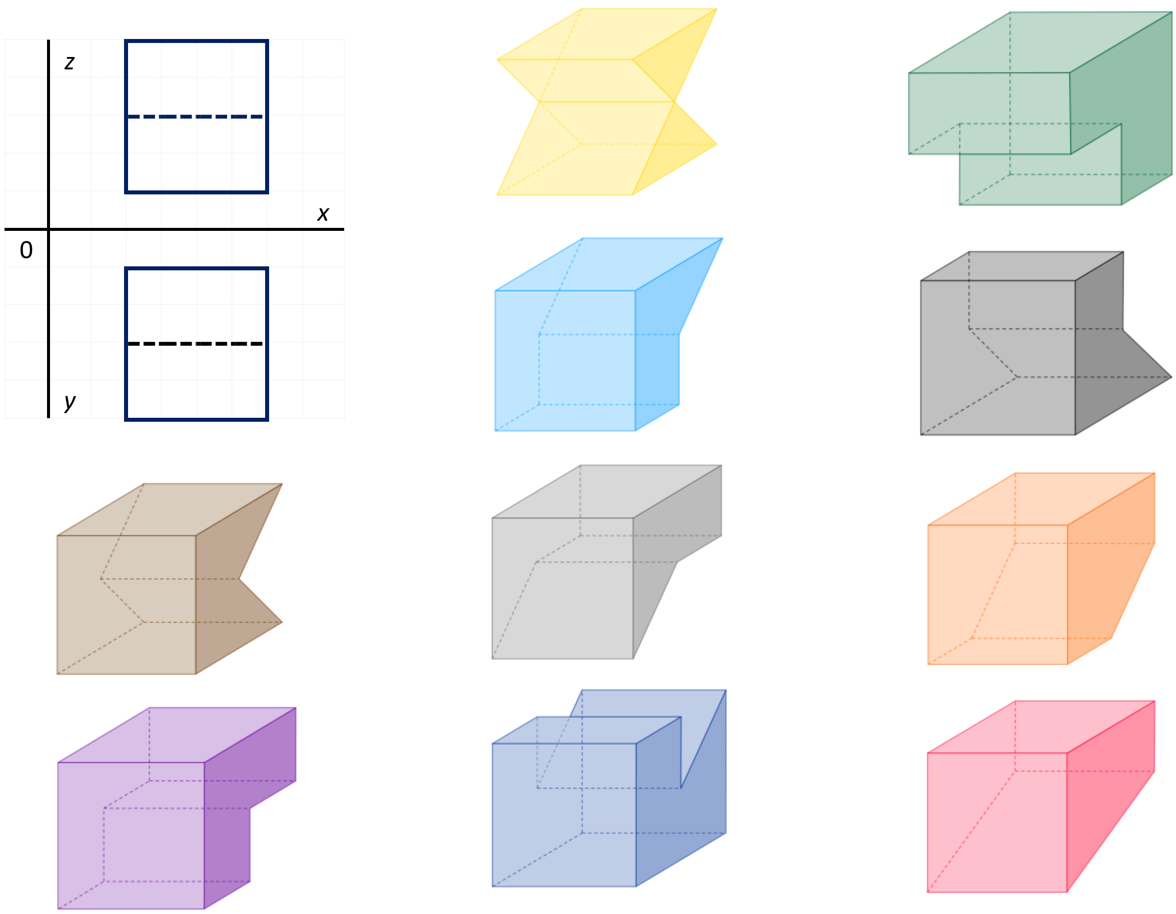


For this problem, we have several solutions. These are few of them:



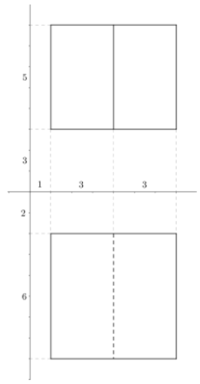
Of course, if we add the side view, then we obtain the unique solution.

An example of the student's task: select the objects to which the given top and front orthogonal view correspond:



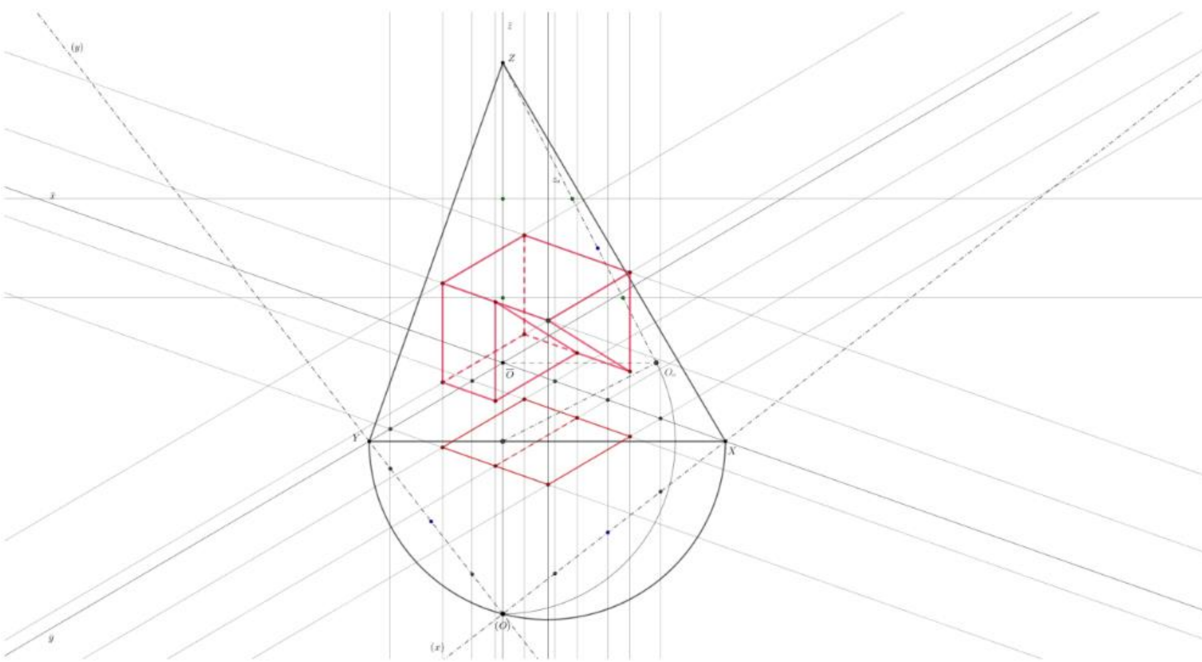
Axonometric projection: construction of an angular object in GeoGebra

A more complex task is the construction of the orthogonal axonometric image of a non-regular angular object, which is given with its front and top view of orthogonal projections:

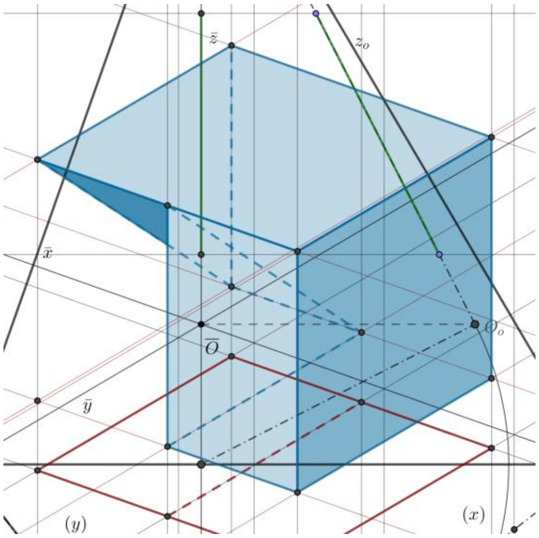


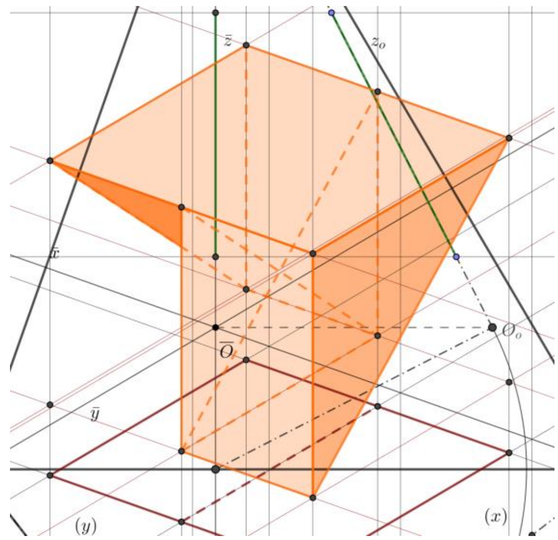
Beside the technique of the construction, we also emphasize on how accurately one can determine or visualize the true shape of the object.
Once the object is defined considering the visibility of its edges, *i.e.* its sides in 3D image, the construction of its orthogonal axonometry may begin.

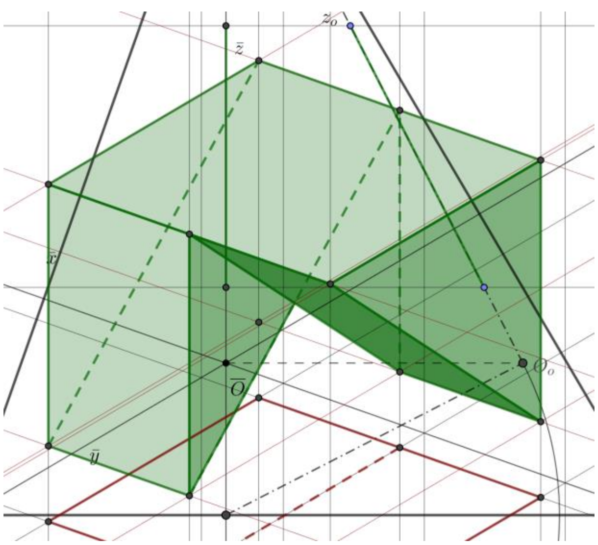
The step-by-step construction in GeoGebra is done in detail, and this is a one of the solution:



Other solutions are:

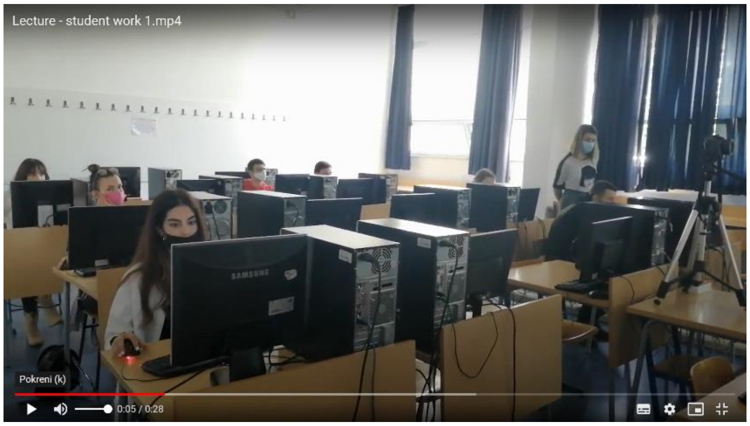


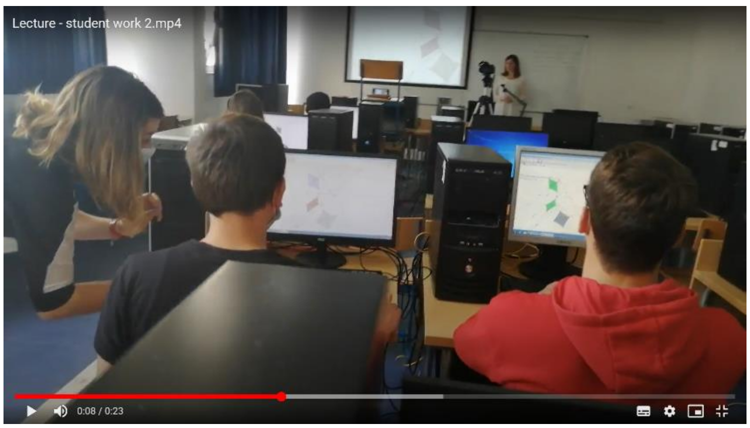




GeoGebra in the classroom

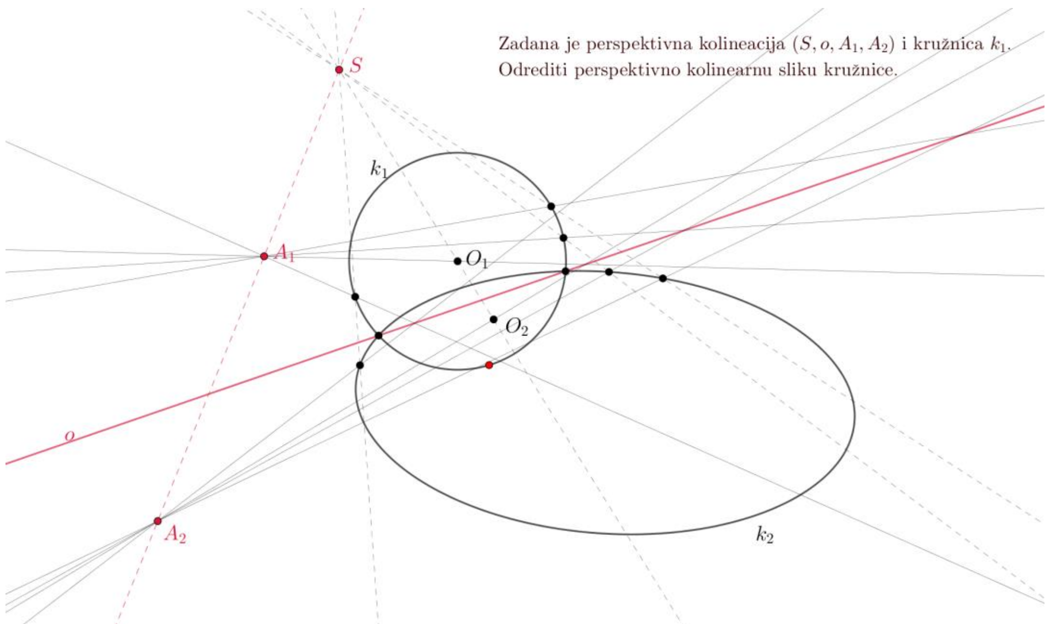
The use of computer programs in teaching, in addition to classical lectures, has proven to be successful and effective, both for students and for us, professors. Students are more actively involved in classes and show greater interest and motivation. In these pictures we can see students constructing a polygon in Monge’s projection, using GeoGebra:

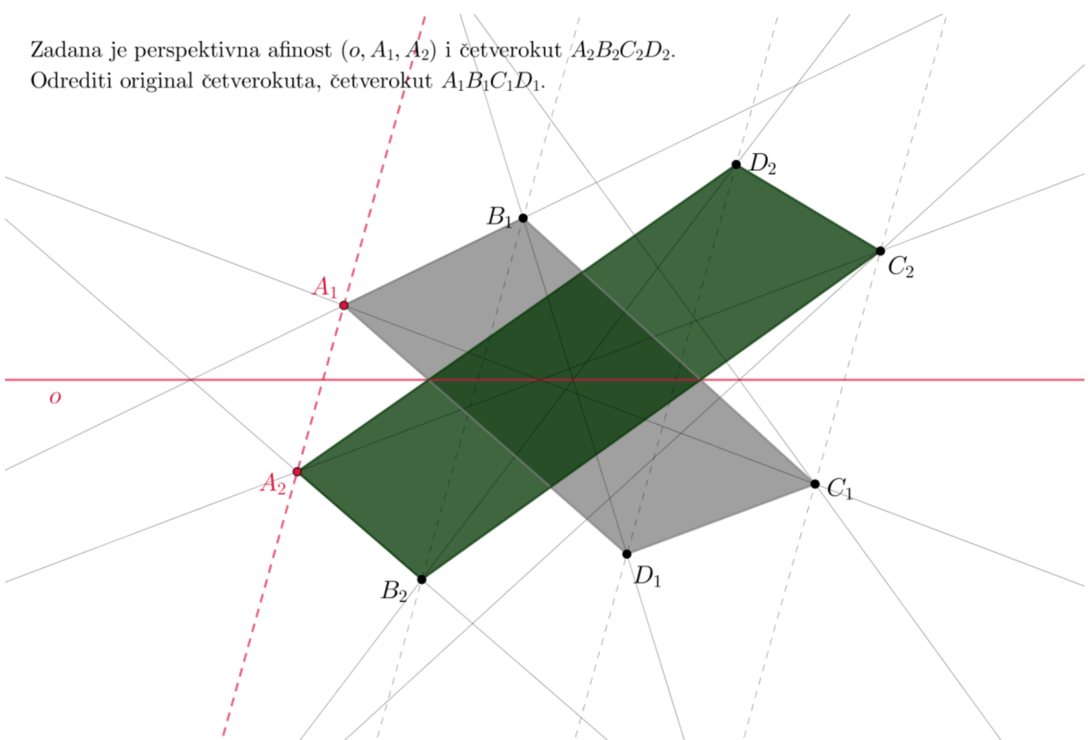




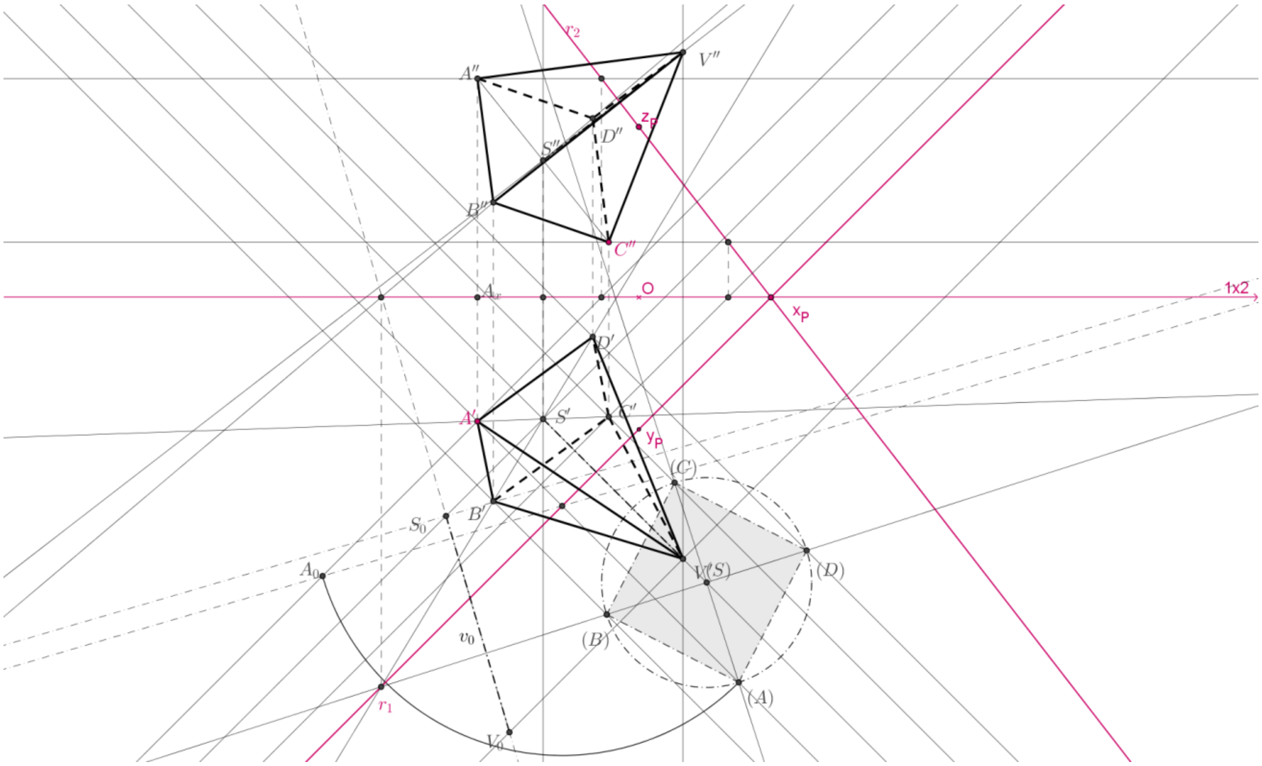
Finally, we give an overview of the application of different computer programs (CAD software and software for dynamic geometry) in solving geometric problems.

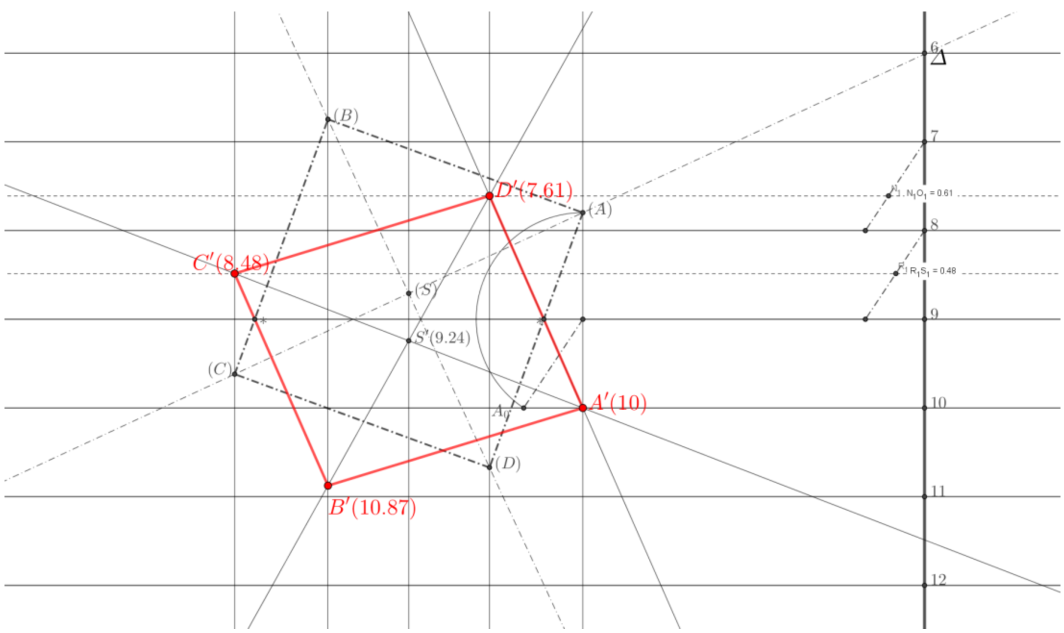
Perspective collineation and affinity



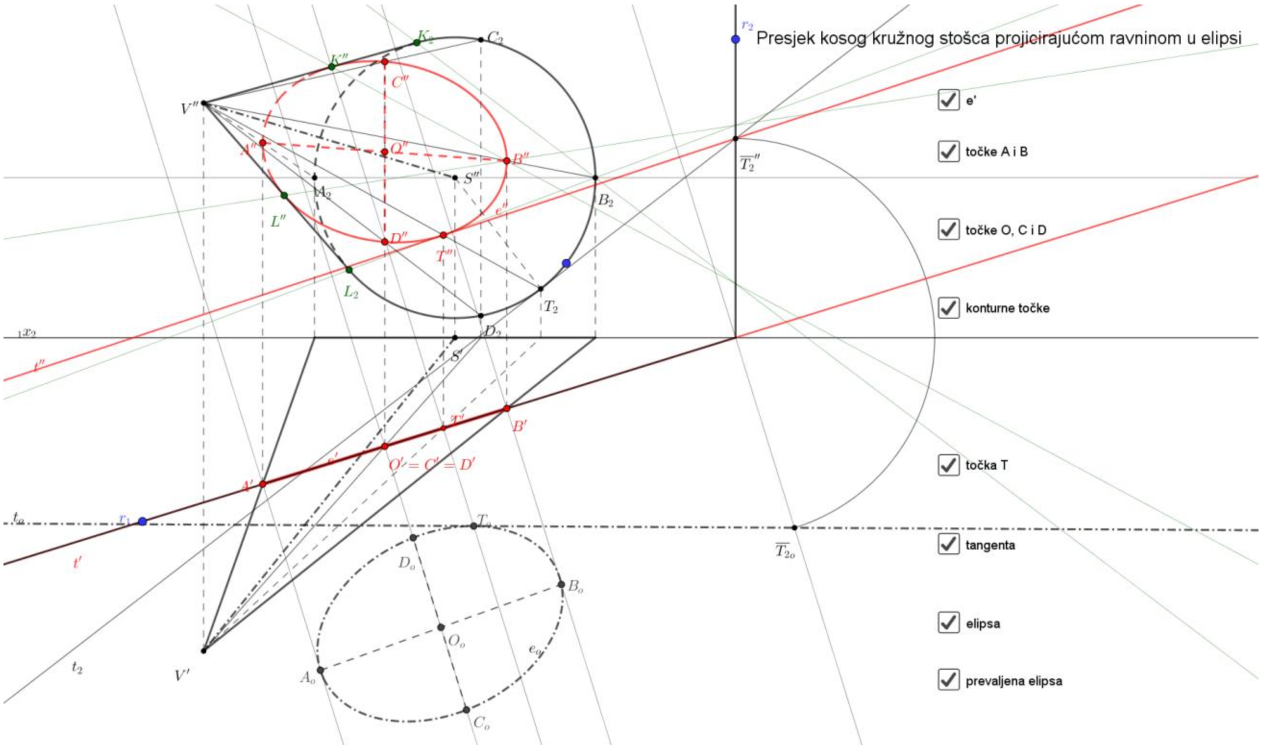


Monge’s and orthogonal projections

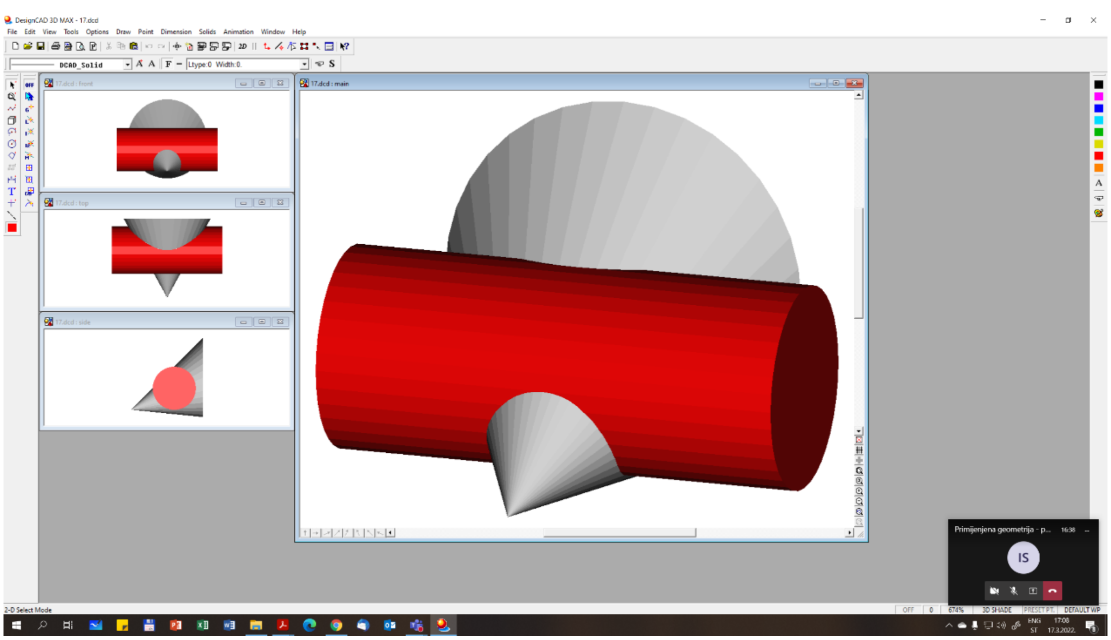


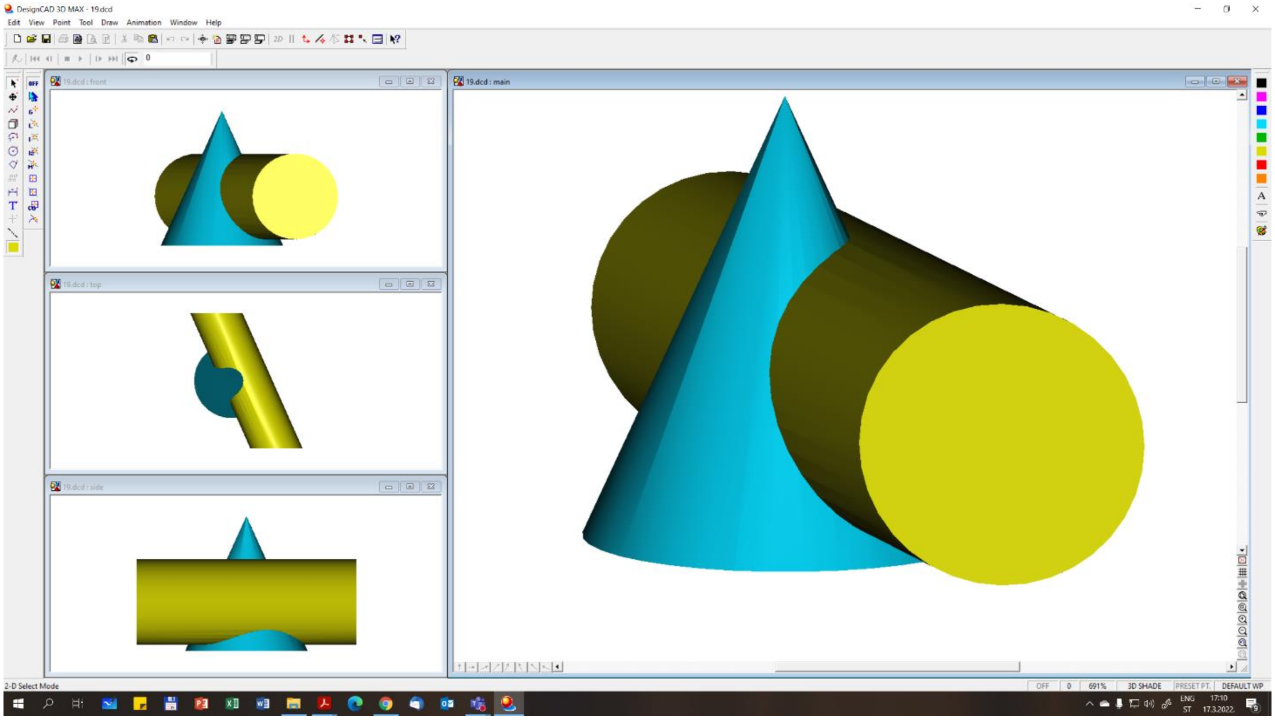


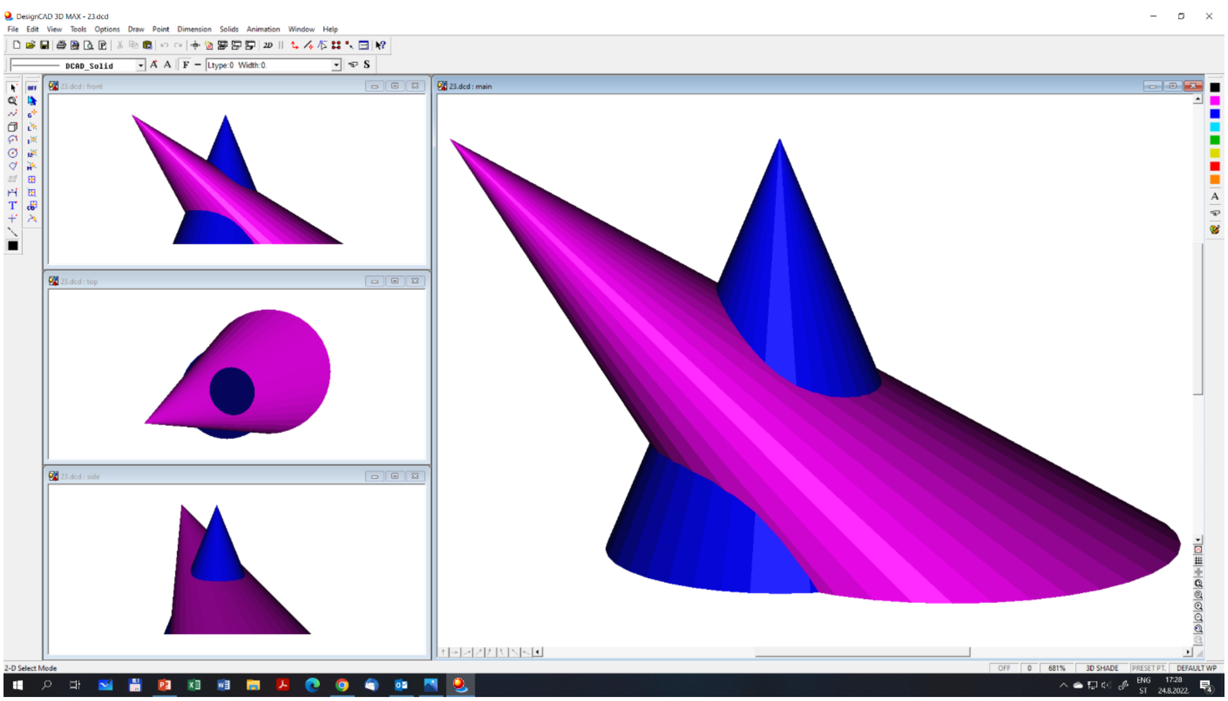
Intersection of the conus and the plane

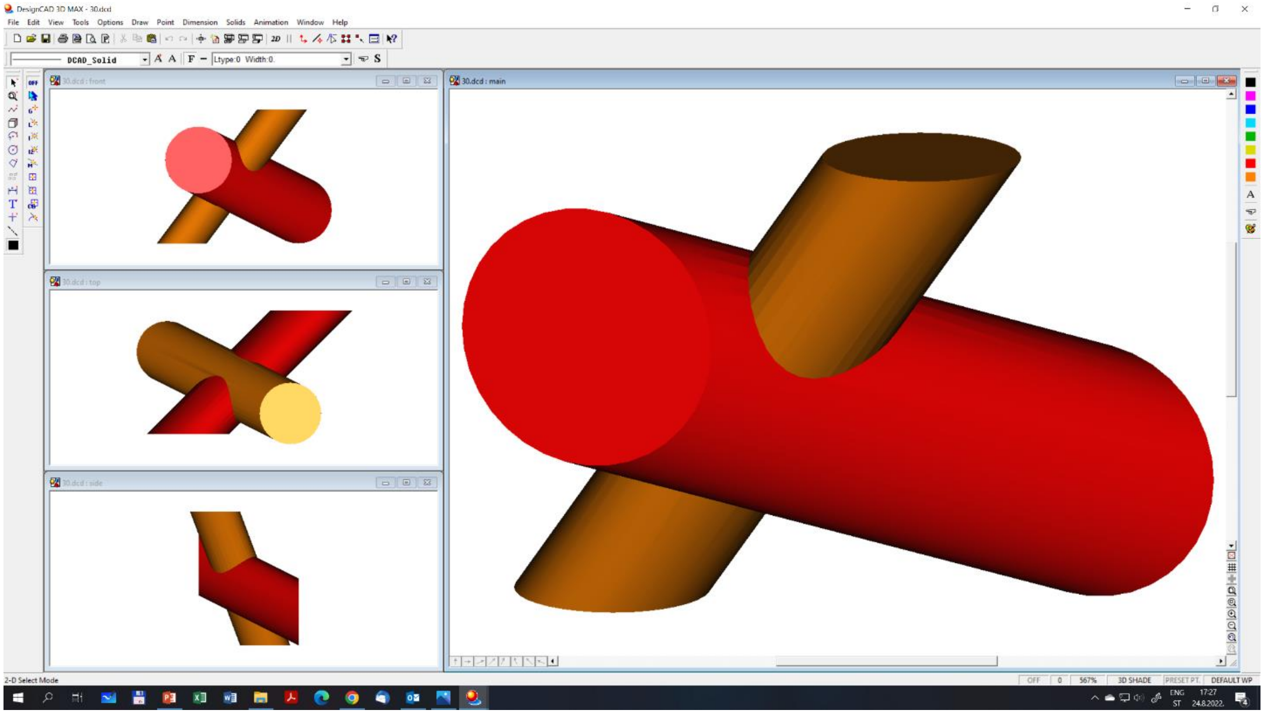


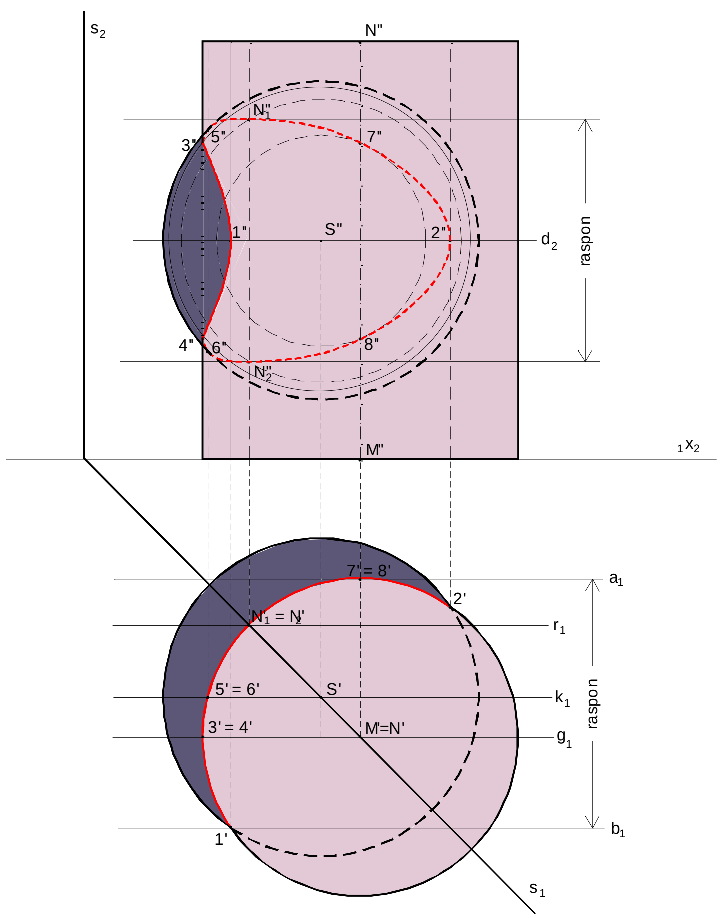
Intersection of solids



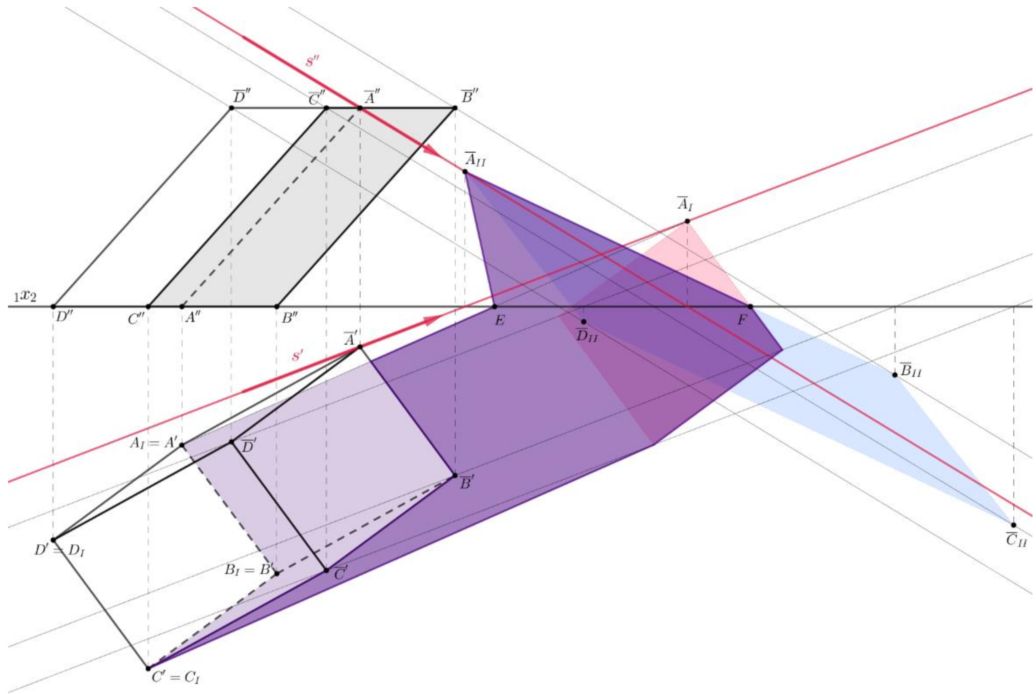

DesignCAD 3D MAX - 19.dcd
Edit View Point Tool Draw Animation Window Help
19.dcd : main
2-D Select Mode
3D SHADE
DEFAULT WP

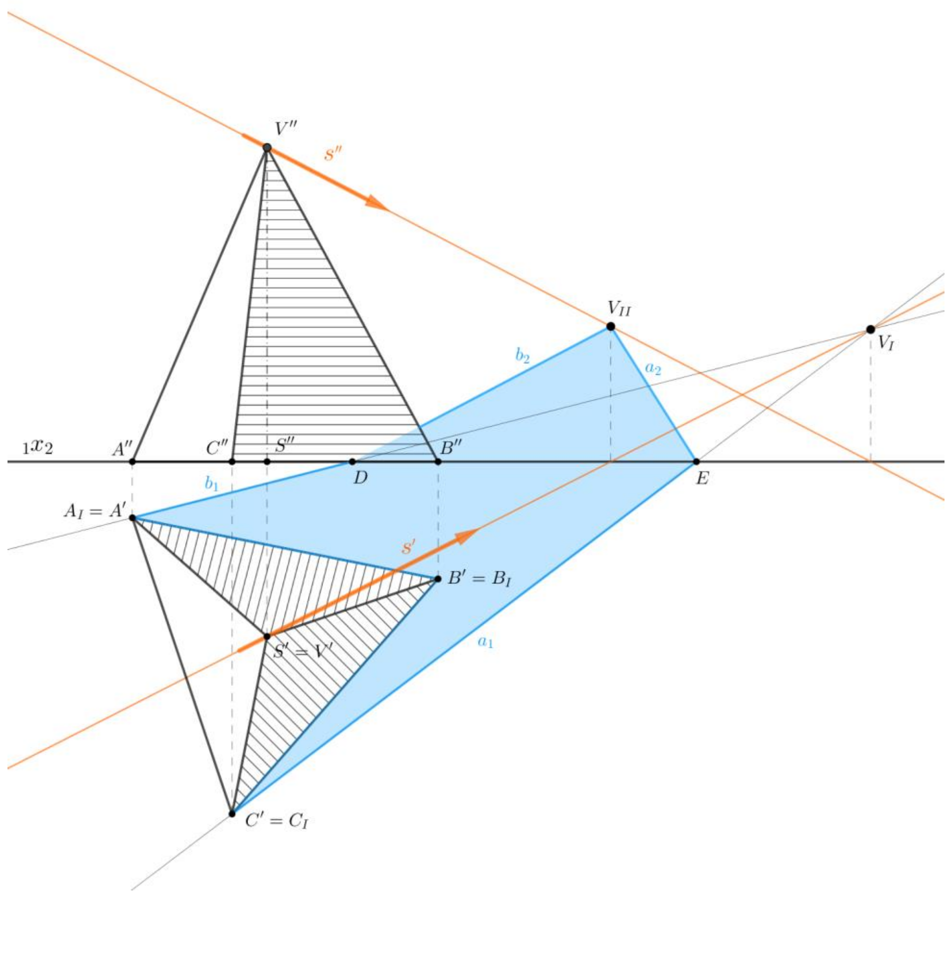

DesignCAD 3D MAX - 23.dcd
File Edit View Tools Options Draw Point Dimension Solids Animation Window Help
DCAD_Solid
Ltype:0 Width:0
23.dcd : main
2-D Select Mode
3D SHADE
DEFAULT WP

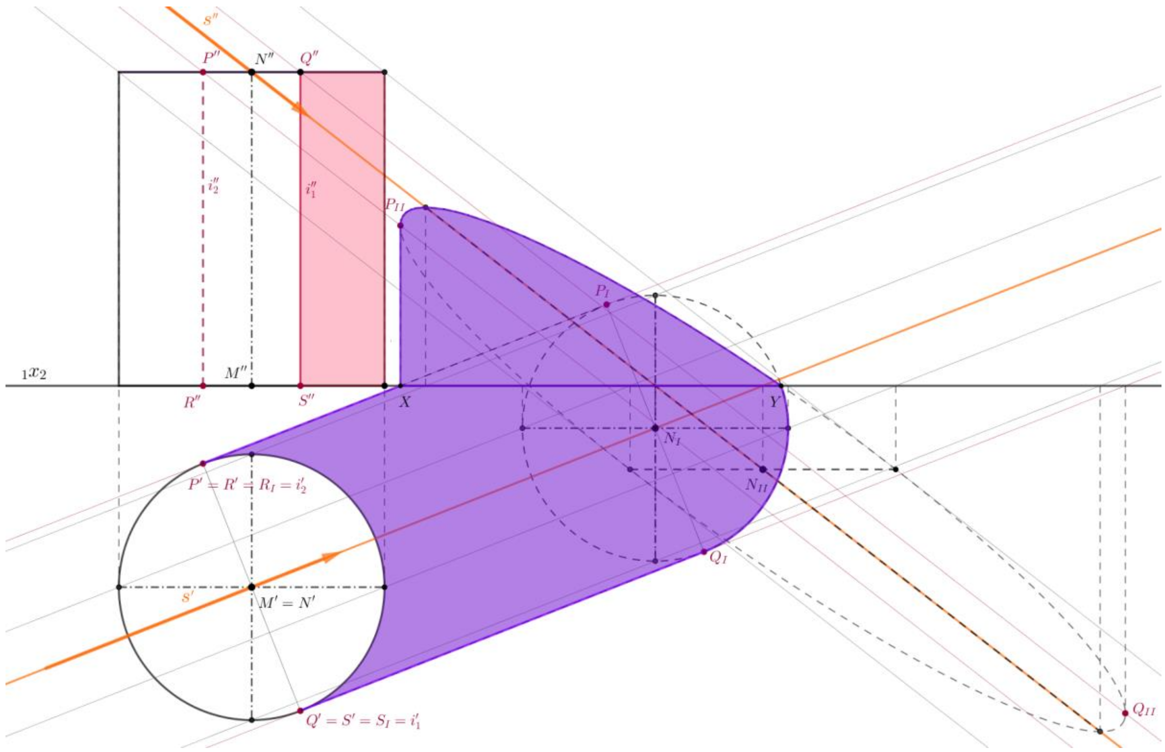

DesignCAD 3D MAX - 30.dcd
File Edit View Tools Options Draw Point Dimension Solids Animation Window Help
30.dcd : main
2-D Select Mode


s2
N"
N1"
5"
7"
3"
1"
S"
2"
d2
raspon
4"
6"
8"
N2"
M"
1x2
7' = 8'
a1
2'
N'1 = N'2
r1
5' = 6'
S'
k1
3' = 4'
M'=N'
g1
raspon
b1
1'
s1

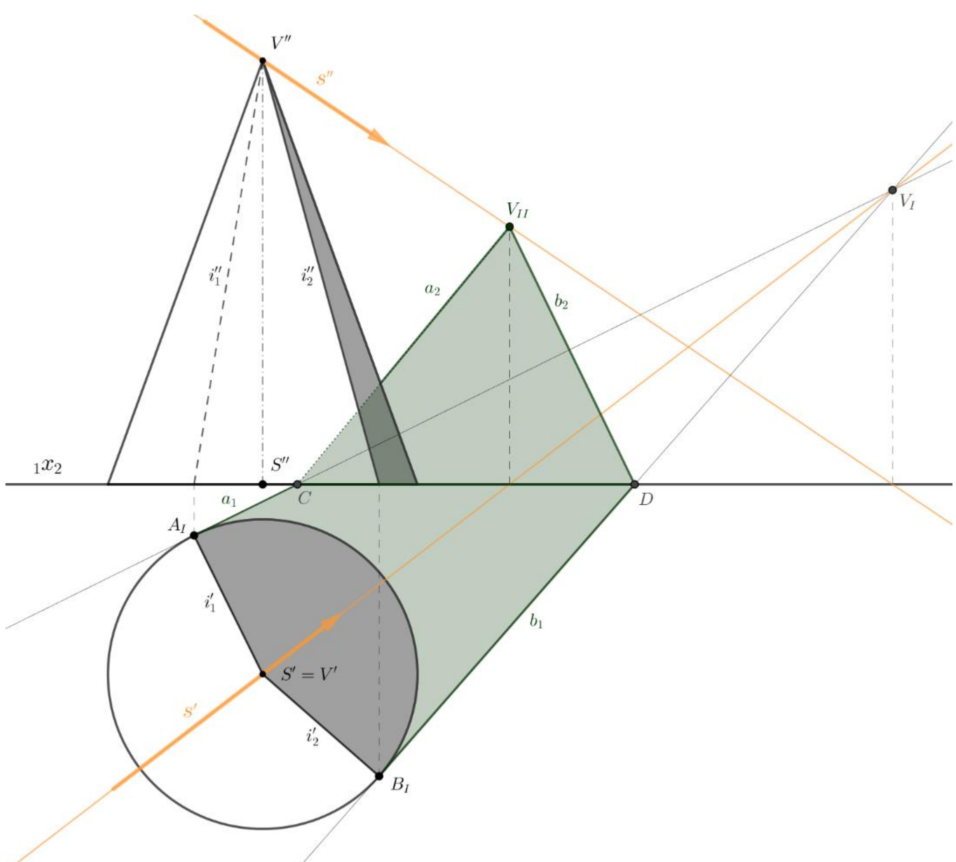
Shades and shadows



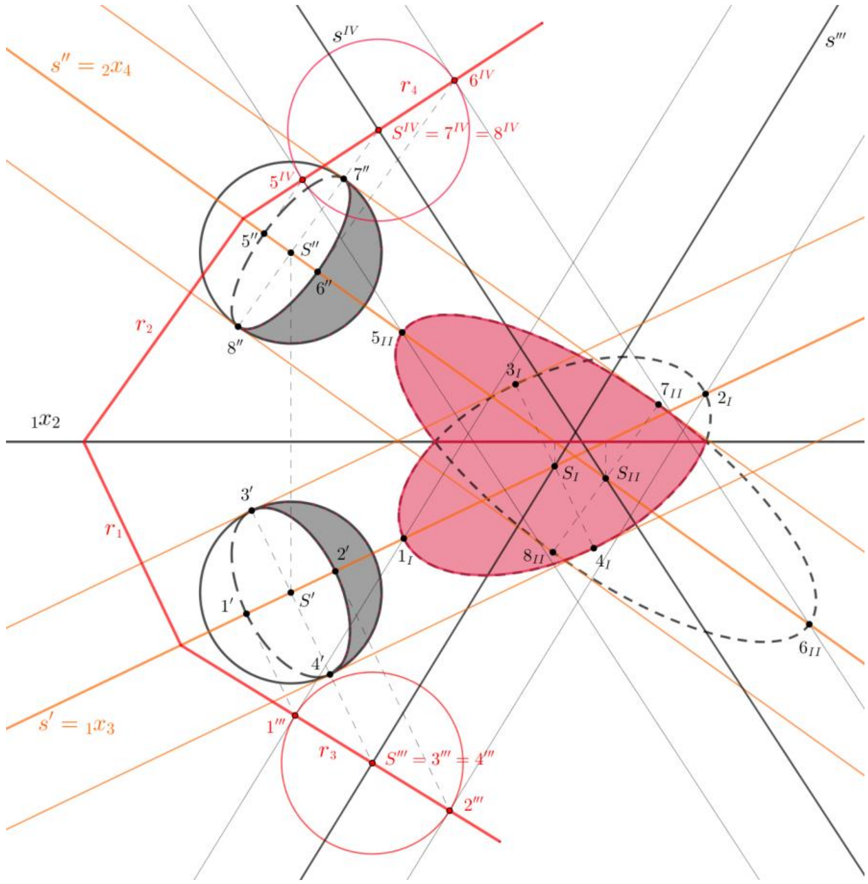




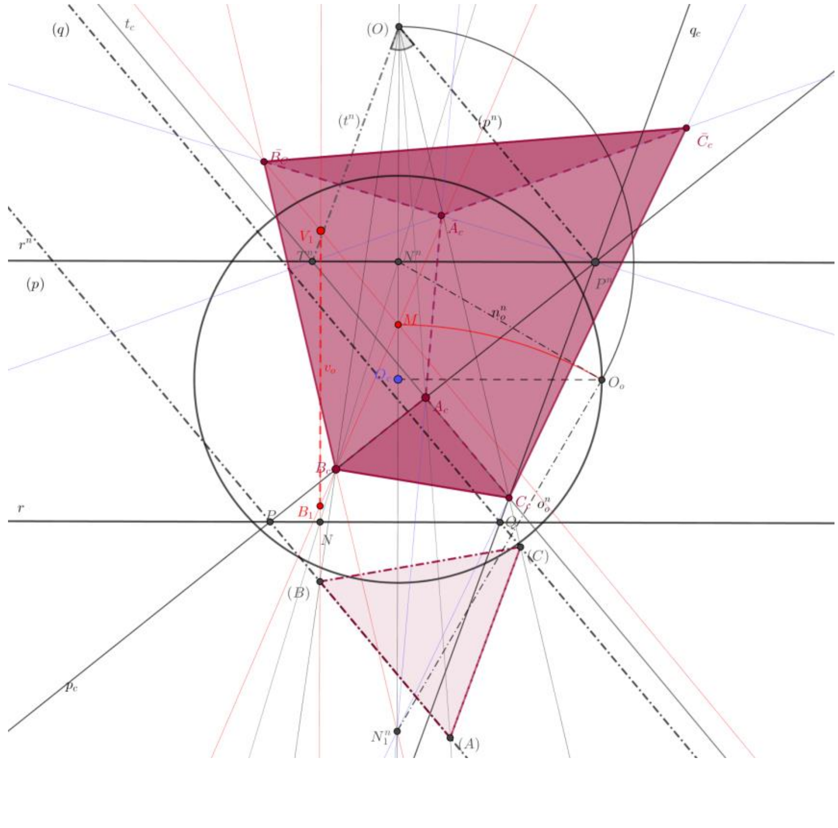
s''
P''
N''
Q''
i_2''
i_1''
P_{II}
P_I
${}_1x_2$
M''
R''
S''
X
Y
N_I
N_{II}
$P' = R' = R_I = i_2'$
Q_I
s'
$M' = N'$
$Q' = S' = S_I = i_1'$
Q_{II}

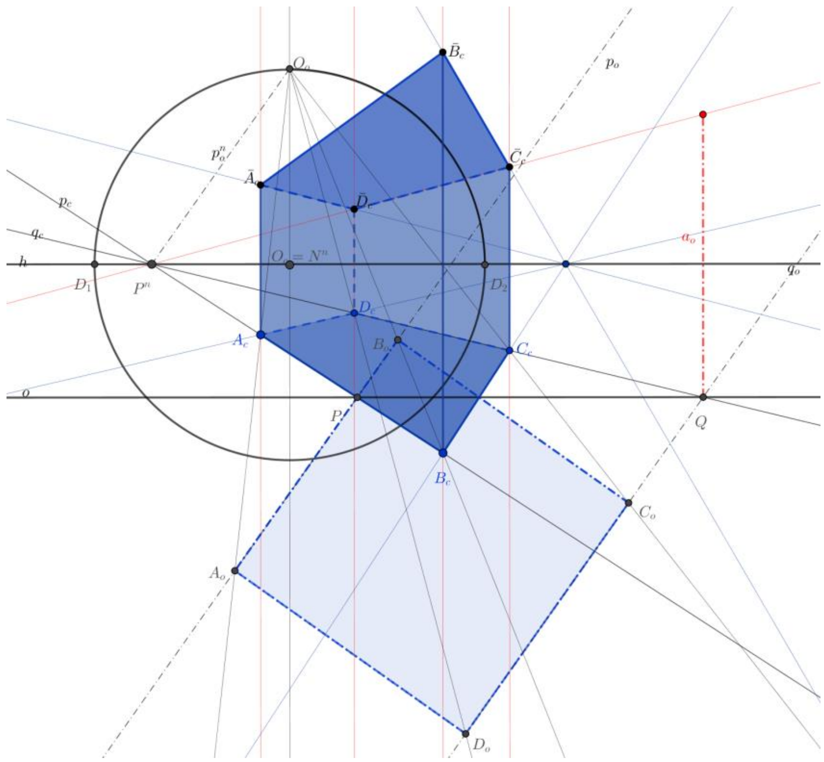


V''
s''
V_{II}
V_I
a_2
b_2
i_1''
i_2''
${}_1x_2$
S''
a_1
C
D
A_I
i_1'
b_1
$S' = V'$
s'
i_2'
B_I



Perspective




O_o
$\bar{B}_c$
p_o
p_o^n
$\bar{A}_c$
$\bar{C}_c$
$\bar{D}_c$
p_c
q_c
a_o
h
$O_c = N^n$
D_1
P^n
D_2
q_o
D_c
A_c
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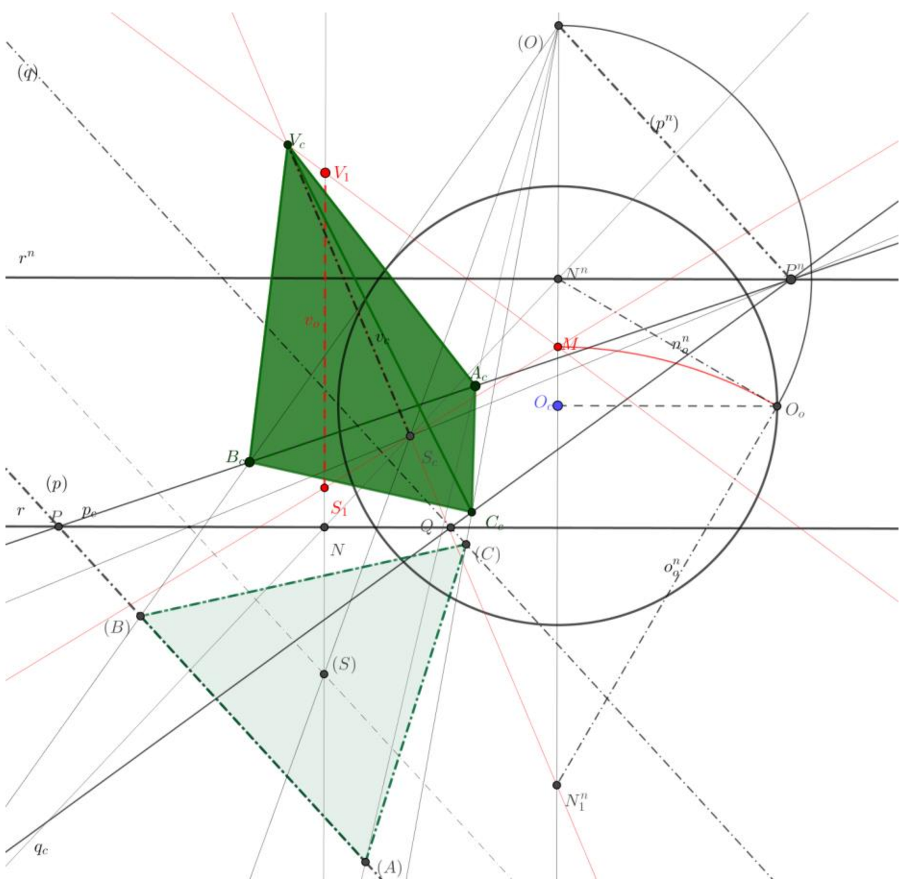
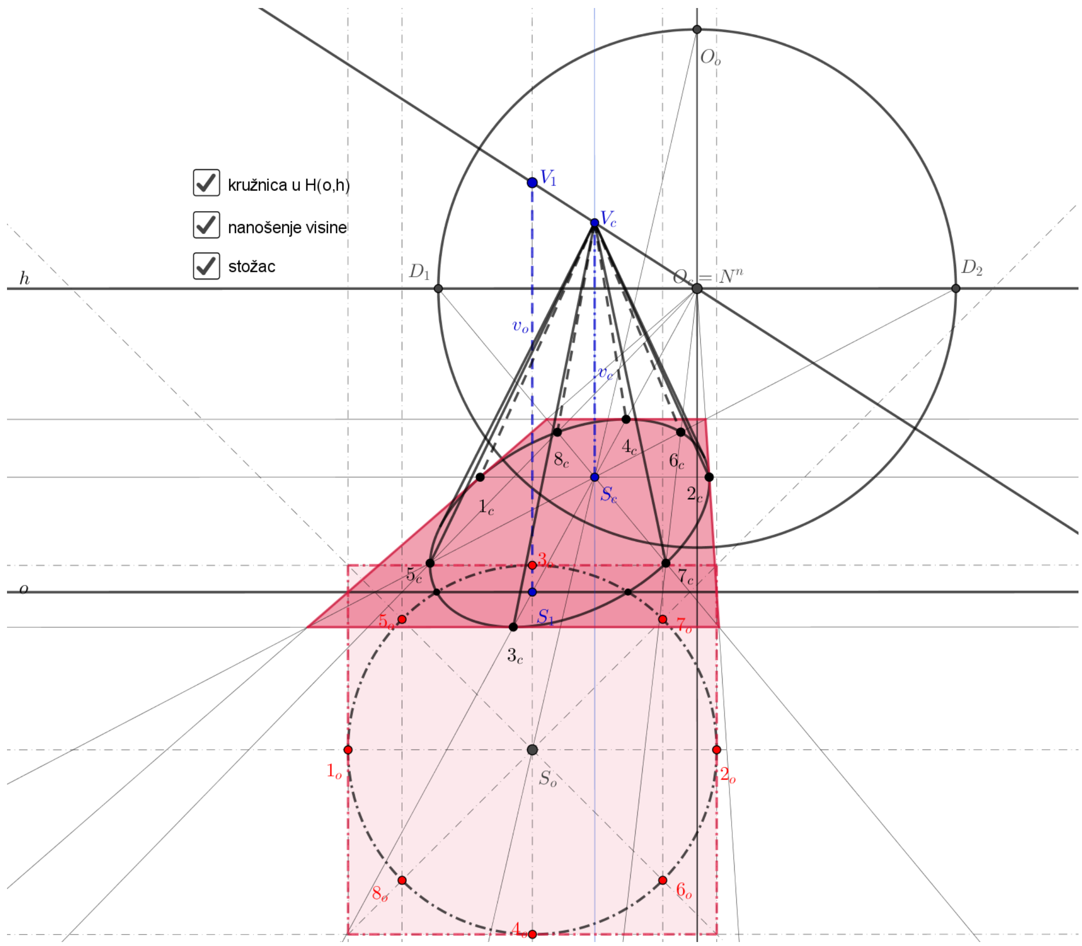
P
Q
B_c
C_o
A_o
D_o


(O)
(q)
(p^n)
V_c
V_1
r^n
N^n
P^n
v_o
v_c
M
n_o^n
A_c
O_c
O_o
B_c
S_c
(p)
S_1
r
P
p_c
Q
C_c
N
(C)
o_o^n
(B)
(S)
N_1^n
q_c
(A)



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- KoG, Scientific and Professional Journal of Croatian Society for Geometry and Graphics, No. 7, Zagreb, (2003)
- Photographs: online resources and private albums